2015 年 7 月 10 日 (2015 年 7 月 14 日訂正)
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微分積分学第一講義資料 9

お知らせ

- 授業評価へのご協力お願いいたします．7 月 8 日 10 時現在 17/114．目標 90/114.
- 次週 7 月 17 日に中間試験を行います．すでに予告をしておりますが，聞いていないというかたは講義 web ページ，OCW より「中間試験予告」の用紙を入手しておいて下さい.

前回の補足

- 陰関数定理と陰関数の微分公式については，今回少し補足します.
- 平面のラプラシアンを極座標で表す計算について，質問がありましたので，きちんと計算を書いておきます：
  変数変換

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{1}$$

の微分（ヤコビ行列）は

$$J := \begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

となるから，(1) を (r,θ) について解いて得られる $r = r(x,y)$, $\theta = \theta(x,y)$ のヤコビ行列は

$$\begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix} = J^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{pmatrix} \tag{2}$$

となる．したがって $f(x,y)$ を変数変換 (1) で (r,θ) の関数として考えると，チェイン・ルールから

$$\begin{aligned} \frac{\partial f}{\partial x} &= r_x\frac{\partial f}{\partial r} + \theta_x\frac{\partial f}{\partial \theta} = \cos\theta\frac{\partial f}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial f}{\partial \theta} \\ \frac{\partial f}{\partial y} &= r_y\frac{\partial f}{\partial r} + \theta_y\frac{\partial f}{\partial \theta} = \sin\theta\frac{\partial f}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial f}{\partial \theta} \end{aligned}$$

とくに f は何でもよいので,

$$\frac{\partial *}{\partial x} = r_x\frac{\partial *}{\partial r} + \theta_x\frac{\partial *}{\partial \theta} = \cos\theta\frac{\partial *}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial *}{\partial \theta} \tag{3}$$

$$\frac{\partial *}{\partial y} = r_y\frac{\partial *}{\partial r} + \theta_y\frac{\partial *}{\partial \theta} = \sin\theta\frac{\partial *}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial *}{\partial \theta} \tag{4}$$

とかける．いま

$$\frac{\partial f}{\partial x} = \cos\theta\frac{\partial f}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial f}{\partial \theta}$$

なので，この両辺を x で偏微分すると（上の式を (3) の $*$ として）

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\cos\theta\frac{\partial f}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial f}{\partial \theta}\right) \\
&= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial f}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial f}{\partial \theta}\right) - \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial f}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial f}{\partial \theta}\right) \\
&= \cos\theta\left[\cos\theta\frac{\partial}{\partial r}\frac{\partial f}{\partial r} - \left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\sin\theta\frac{\partial f}{\partial \theta} + \frac{1}{r}\sin\theta\frac{\partial}{\partial r}\frac{\partial f}{\partial \theta}\right]\right\} \\
&\quad - \frac{1}{r}\sin\theta\left[\left\{\frac{\partial}{\partial \theta}(\cos\theta)\frac{\partial f}{\partial r} + \cos\theta\frac{\partial}{\partial \theta}\frac{\partial f}{\partial r}\right\} - \left\{\frac{1}{r}\frac{\partial}{\partial \theta}(\sin\theta)\frac{\partial f}{\partial \theta} + \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta}\frac{\partial f}{\partial \theta}\right\}\right] \\
&= \cos\theta\left[\cos\theta\frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\sin\theta\frac{\partial f}{\partial \theta} - \frac{1}{r}\sin\theta\frac{\partial^2 f}{\partial r\partial \theta}\right] \\
&\quad - \frac{1}{r}\sin\theta\left[-\sin\theta\frac{\partial f}{\partial r} + \cos\theta\frac{\partial^2 f}{\partial \theta\partial r} - \frac{1}{r}\cos\theta\frac{\partial f}{\partial \theta} - \frac{1}{r}\sin\theta\frac{\partial^2 f}{\partial \theta^2}\right] \\
&= \cos^2\theta\frac{\partial^2 f}{\partial r^2} - \frac{2}{r}\cos\theta\sin\theta\frac{\partial^2 f}{\partial r\partial \theta} + \frac{1}{r^2}\sin^2\theta\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r}\sin^2\theta\frac{\partial f}{\partial r} + \frac{2}{r^2}\cos\theta\sin\theta\frac{\partial f}{\partial \theta}, \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\sin\theta\frac{\partial f}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial f}{\partial \theta}\right) \\
&= \sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial f}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r}\cos\theta\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial f}{\partial \theta}\right) \\
&= \sin\theta\left[\sin\theta\frac{\partial}{\partial r}\frac{\partial f}{\partial r} + \left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\cos\theta\frac{\partial f}{\partial \theta} + \frac{1}{r}\cos\theta\frac{\partial}{\partial r}\frac{\partial f}{\partial \theta}\right]\right\} \\
&\quad + \frac{1}{r}\cos\theta\left[\left\{\frac{\partial}{\partial \theta}(\sin\theta)\frac{\partial f}{\partial r} + \sin\theta\frac{\partial}{\partial \theta}\frac{\partial f}{\partial r}\right\} + \left\{\frac{1}{r}\frac{\partial}{\partial \theta}(\cos\theta)\frac{\partial f}{\partial \theta} + \frac{1}{r}\cos\theta\frac{\partial}{\partial \theta}\frac{\partial f}{\partial \theta}\right\}\right] \\
&= \sin\theta\left[\sin\theta\frac{\partial^2 f}{\partial r^2} - \frac{1}{r^2}\sin\theta\frac{\partial f}{\partial \theta} + \frac{1}{r}\sin\theta\frac{\partial^2 f}{\partial r\partial \theta}\right] \\
&\quad + \frac{1}{r}\cos\theta\left[\cos\theta\frac{\partial f}{\partial r} + \sin\theta\frac{\partial^2 f}{\partial \theta\partial r} - \frac{1}{r}\sin\theta\frac{\partial f}{\partial \theta} + \frac{1}{r}\cos\theta\frac{\partial^2 f}{\partial \theta^2}\right] \\
&= \sin^2\theta\frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\cos\theta\sin\theta\frac{\partial^2 f}{\partial r\partial \theta} + \frac{1}{r^2}\cos^2\theta\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r}\cos^2\theta\frac{\partial f}{\partial r} - \frac{2}{r^2}\cos\theta\sin\theta\frac{\partial f}{\partial \theta}.
\end{aligned}
$$

したがって

$$
\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}.
$$