1 A review of the surface theory.

The Euclidean space. We denote by $\mathbb{R}^3$ the *Euclidean 3-space* with *inner product* " $\cdot$ ", that is,

$$\boldsymbol{x}\cdot\boldsymbol{y} := x_1y_1+x_2y_2+x_3y_3 = {}^t\boldsymbol{x}\boldsymbol{y}, \qquad \left(\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \ \boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right).$$

Here, we consider vectors in $\mathbb{R}^3$ as column vectors and "^{t} " denotes the *transposition*. The *Euclidean distance* d of $\mathbb{R}^3$ is defined as $d(\boldsymbol{x},\boldsymbol{y}) := |\boldsymbol{y}-\boldsymbol{x}|$, where $|\boldsymbol{v}| := \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}$. An *isometry* of $\mathbb{R}^3$, that is, a transformation $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ preserving the Euclidean distance, has the following form:

$$(1.1) \qquad F(\boldsymbol{x}) = P\boldsymbol{x}+\boldsymbol{b} \qquad P \in \mathrm{O}(3), \quad \boldsymbol{b} \in \mathbb{R}^3.$$

Here, we denote

$$\mathrm{O}(3) = \text{the set of } 3\times 3 \text{ orthogonal matrices},$$
$$\mathrm{SO}(3) = \{P \in \mathrm{O}(3) \mid \det P = 1\}.$$

A basis $\{\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3\}$ of $\mathbb{R}^3$ is said to be *positive* (resp. *negative*) if $\det(\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3)$ is positive (resp. negative). The triple of the column vectors of a matrix in $\mathrm{O}(3)$ (resp. $\mathrm{SO}(3)$) forms an *orthonormal basis* (resp. a *positive orthonormal basis*).

An isometry as in (1.1) is called an *orientation preserving isometry* (resp. an *orientation reversing isometry*) if $A \in \mathrm{SO}(3)$ (resp. $A \in \mathrm{O}(3)\setminus\mathrm{SO}(3)$).

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Immersed surfaces. Consider a smooth[1] 2-*manifold* Σ and a smooth map

$$(1.2) \qquad f\colon \Sigma \ni \mathrm{P} \longmapsto f(\mathrm{P}) = {}^t\big(x(\mathrm{P}),y(\mathrm{P}),z(\mathrm{P})\big) \in \mathbb{R}^3.$$

Then each component of f is a smooth function defined on Σ. For each point $\mathrm{P} \in \Sigma$, we define a liner map $df_{\mathrm{P}}\colon T_{\mathrm{P}}\Sigma \to \mathbb{R}^3$ as

$$(1.3) \qquad df_{\mathrm{P}}(X) := {}^t\big(dx_{\mathrm{P}}(X), dy_{\mathrm{P}}(X), dz_{\mathrm{P}}(X)\big) \quad (X \in T_{\mathrm{P}}\Sigma),$$

which is called the *differential* of the map f at P, where dx, dy and dz are the usual differential of smooth functions.

Definition 1.1. A map f as in (1.2) is immersive at P if the map $df_{\mathrm{P}}\colon T_{\mathrm{P}}\Sigma \to \mathbb{R}^3$ is injective. Moreover, f is said to be an *immersion* if f is immersive at all $\mathrm{P} \in \Sigma$. In this lecture, an immersion as (1.2) is called an *immersed surface*.

Let $(U,(u,v))$ be a local coordinate chart of Σ at P. Then f in (1.2) is considered as an $\mathbb{R}^3$-valued function of (u,v), and

$$df\left(\frac{\partial}{\partial u}\right) = f_u = {}^t(x_u,y_u,z_u), \qquad df\left(\frac{\partial}{\partial v}\right) = f_v.$$

In particular, the image of $df_{(u,v)}$ is spanned by $f_u(u,v)$, $f_v(u,v)$. Thus, we have

Proposition 1.2. *The map $f\colon U \ni (u,v) \mapsto f(u,v) \in \mathbb{R}^3$ is an immersion if and only if $f_u(u,v)$ and $f_v(u,v)$ are linearly independent for each $(u,v) \in U$.*

[1] We use the word smooth for "of class C^∞", in this lecture.

Change of Parameters. Let $f\colon \Sigma \to \mathbb{R}^3$ be an immersion of a 2-manifold Σ. Take local coordinate charts $(U,(u,v))$ and $(V,(\xi,\eta))$ on a neighborhood of $\mathrm{P} \in \Sigma$. Then the change of coordinates is a pair of smooth functions

$$u = u(\xi,\eta), \qquad v = v(\xi,\eta) \tag{1.4}$$

such that (ξ,η) in V and $\bigl(u(\xi,\eta), v(\xi,\eta)\bigr) \in U$ corresponds to the same point of Σ. We write its *Jacobian matrix*

$$J := \begin{pmatrix} u_\xi & u_\eta \\ v_\xi & v_\eta \end{pmatrix}, \tag{1.5}$$

such that $\det J$ does not vanish everywhere. We can write

$$du = u_\xi d\xi + u_\eta d\eta, \qquad dv = v_\xi d\xi + v_\eta d\eta. \tag{1.6}$$

The coordinate change (1.4) is said to be *orientation preserving* (resp. *orientation reversing*) if $\det J$ is positive (resp. negative). Two coordinate charts $(U,(u,v))$ and $(V,(\xi,\eta))$ are *compatible* if the change of coordinates is orientation preserving.

Definition 1.3. A manifold Σ is *orientable* if there exists an atlas $\mathcal{A} = \{(U_j,(u_j,v_j))\}$ of Σ such that each charts in $\mathcal{A}$ are compatible. Such a choice of atlas is called the *orientation* of Σ. The manifold Σ is called *oriented* if one orientation is specified. In this case, a coordinate chart $(U,(u,v))$ is said to be *compatible to the orientation* if it is compatible to one of the chart in the fixed orientation.

For the sake of simplicity, we consider only oriented manifolds in this lecture.

The unit normal vector.

Definition 1.4. The *unit normal vector field* ν of an immersion $f\colon \Sigma \to \mathbb{R}^3$ on a domain $U \subset \Sigma$ is a smooth map $\nu\colon \Sigma \supset U \to \mathbb{R}^3$ such that

$$df_{\mathrm{P}}(T_{\mathrm{P}}\Sigma) \perp \nu(\mathrm{P}), \qquad |\nu(\mathrm{P})| = 1 \tag{1.7}$$

hold for all $\mathrm{P} \in U$.

Remark 1.5. For a local coordinate chart $(U,(u,v))$ of Σ,

$$\nu(u,v) := \frac{f_u(u,v) \times f_v(u,v)}{|f_u(u,v) \times f_v(u,v)|} \tag{1.8}$$

is a unit normal vector field on U of $f\colon \Sigma \to \mathbb{R}^3$, where "$\times$" denotes the *vector product* or the *outer product* of $\mathbb{R}^3$.

Since (1.8) does not depend on the *orientation preserving* change of coordinates, one can find globally defined unit normal vector field of f if Σ is oriented.

The first fundamental form Let $f\colon \Sigma \to \mathbb{R}^3$ be an immersion and $(U,(u,v))$ a coordinate chart of Σ. The *first fundamental form* (or the *induced metric*) of f is defined as

$$\begin{aligned} ds^2 &:= df \cdot df = (f_u\,du + f_v\,dv)\cdot(f_u\,du + f_v\,dv) \\ &= E\,du^2 + 2F\,du\,dv + G\,dv^2, \\ &\qquad E = f_u \cdot f_u, \quad F = f_u \cdot f_v, \quad G = f_v \cdot f_v. \end{aligned} \tag{1.9}$$

The functions E, F, G in (u,v) are called the *entries* of the first fundamental form. Let $(V,(\xi,\eta))$ be another coordinate chart of Σ, and consider a change of coordinates as in (1.4). Then we have, by the chain-rule,

$$\begin{pmatrix} \widetilde{E} & \widetilde{F} \\ \widetilde{F} & \widetilde{G} \end{pmatrix} = {}^tJ \begin{pmatrix} E & F \\ F & G \end{pmatrix} J, \qquad (1.10)$$

$$\text{where} \quad \widetilde{E} = f_\xi \cdot f_\xi, \quad \widetilde{F} = f_\xi \cdot f_\eta, \quad \widetilde{G} = f_\eta \cdot f_\eta,$$

where J is the Jacobian matrix (1.5) of the coordinate change. Moreover, the first fundamental form does not depend of the choice of coordinate charts by virtue of (1.6):

$$E\,du^2 + 2F\,du\,dv + G\,dv^2 = \widetilde{E}\,d\xi^2 + 2\widetilde{F}d\xi, d\eta + \widetilde{G}\,d\eta^2.$$

By the Schwartz inequality, we have

Lemma 1.6. $EG - F^2 > 0$.

Lemma 1.7. *Let $f\colon \Sigma \to \mathbb{R}^3$ be an immersion and $(U,(u,v))$ a local coordinate chart. Then*

$$\boldsymbol{e}_1(u,v) := \frac{1}{\sqrt{E}} f_u, \quad \boldsymbol{e}_2(u,v) := \frac{1}{\sqrt{EG-F^2}}\bigl(-F f_u + E f_v\bigr)$$

form an orthonormal system for each $(u,v) \in U$.

The second fundamental form. Let Σ be an oriented manifold, $f\colon \Sigma \to \mathbb{R}^3$ an immersion, and ν the unit normal vector field. The *second fundamental form* is defined as

$$I\!I := -df \cdot d\nu = -(f_u\,du + f_v\,dv) \cdot (\nu_u\,du + \nu_v\,dv), \qquad (1.11)$$

where $(U,(u,v))$ is any local coordinate chart of Σ.

Lemma 1.8. *The second fundamental form is written as*

$$I\!I = L\,du^2 + 2M\,du\,dv + N\,dv^2,$$

$$\textit{where} \quad \begin{cases} L = -f_u \cdot \nu_u = f_{uu} \cdot \nu, \\ M = -f_u \cdot \nu_v = -f_v \cdot \nu_u = f_{uv} \cdot \nu, \\ N = -f_v \cdot \nu_v = f_{vv} \cdot \nu. \end{cases}$$

Proof. By definition,

$$I\!I = -f_u \cdot \nu_u\,du^2 - (f_u \cdot \nu_v + f_v \cdot \nu_u)\,du\,dv - f_v \cdot \nu_v\,dv^2$$

holds. Here, since f_u and f_v are perpendicular to ν,

$$\begin{aligned} -f_u \cdot \nu_v &= -(f_u \cdot \nu)_v + f_{uv} \cdot \nu = f_{uv} \cdot \nu, \\ -f_v \cdot \nu_u &= -(f_v \cdot \nu)_u + f_{vu} \cdot \nu = -f_{uv} \cdot \nu, \dots \end{aligned}$$

□

The functions L, M, N in Lemma 1.8 is called the *entries* of the second fundamental form.

Similar to the first fundamental form, the second fundamental form does not depend of the choice of coordinates:

$$\begin{pmatrix} \widetilde{L} & \widetilde{M} \\ \widetilde{M} & \widetilde{N} \end{pmatrix} = {}^tJ \begin{pmatrix} L & M \\ M & N \end{pmatrix} J, \qquad (1.12)$$

where L, M, N (resp. $\widetilde{L}$, $\widetilde{M}$, $\widetilde{N}$) are the entries of the second fundamental form in uv- (resp. $\xi\eta$-) coordinates, and J is the Jacobian matrix (1.5).

Curvatures. Let $f\colon \Sigma \to \mathbb{R}^3$ be an immersion of an oriented 2-manifold Σ and take the unit normal vector field ν by Remark 1.5. For a local coordinate chart $(U,(u,v))$ on Σ, one can consider a matrix-valued function by Lemma 1.6:

$$A = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}, \tag{1.13}$$

which is called the *Weingarten matrix*. Taking another coordinate chart $(V,(\xi,\eta))$, one can obtain the Weingarten matrix $\widetilde{A}$ with respect to the coordinates (ξ,η). By (1.10) and (1.12)

$$\widetilde{A} = J^{-1}AJ \tag{1.14}$$

holds, where J is the Jacobian matrix as in (1.5). Hence

Lemma 1.9. *The eigenvalues, the determinant, and the trace of the Weingarten matrix* (1.13) *do not depend on the choice of coordinates* (*compatible to the orientation*).

Lemma 1.10. *The eigenvalues of the Weingarten matrix are invariant under orientation preserving isometries.*

Lemma 1.11 (Theorem 8.7 in [1-1])**.** *The eigenvalues of the Weingarten matrix are real valued functions.*

Definition 1.12. The eigenvalues λ_1, λ_2 of the Weingarten matrix A is called the *prinicipal curvatures*. We call the functions

$$K := \det A = \lambda_1\lambda_2, \qquad H := \frac{1}{2}\operatorname{tr} A = \frac{\lambda_1+\lambda_2}{2}, \tag{1.15}$$

the *Gaussian curvature* and the *mean curvature*, respectively.

Example 1.13. An immersion $(u,v) \mapsto (u,v,0)$ represents the xy-plane in $\mathbb{R}^3$. Since $\nu = {}^t(0,0,1)$ is constant, the Weingarten matrix vanishes identically, and then the principal curvatures are zero. In particular, the plane has zero Gaussian curvature.

Example 1.14. An map $f(u,v) = {}^t(\cos u\cos v, \cos u \sin v, \sin u)$ is immersive on $(-\frac{\pi}{2},\frac{\pi}{2}) \times (-\pi,\pi)$, which represents the sphere of radius 1, whose Gaussian curvature is identically 1.

References

[1-1] 梅原雅顕・山田光太郎：曲線と曲面—微分幾何的アプローチ（改訂版），裳華房，2014.

Exercises

1-1[H] Consider a smooth map $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ as

$$f(u,v) = \left(\frac{a\cos u}{\cosh v}, \frac{a\sin u}{\cosh v}, a(v-\tanh v)+bu\right),$$

where $a>0$ and $b \geqq 0$ are constants satisfying $a^2+b^2=1$.

(1) Find a domain $D \subset \mathbb{R}^2$ satisfying
- The restriction $f|_D$ is an immersion.
- $(0,1) \in D$.
- D is maximal among domains satisfying two conditions above.

(2) Compute the Gaussian curvature of f on D.

(3) Draw a picture of the image of $f|_D$ for $(a,b)=(1,0)$.