2 The Gauss and Weingarten formulae

In this section, we consider an immersion $f\colon \mathbb{R}^2 \supset U \to \mathbb{R}^3$ of the domain U in the uv-plane, and let ν be the unit normal vector field as

$$\nu := \frac{f_u \times f_v}{|f_u \times f_v|}.$$

Then, for each $\mathrm{P} = (u, v) \in U$,

$$\mathcal{F}(u,v) := \{f_u(u,v), f_v(u,v), \nu(u,v)\} \tag{2.1}$$

forms a positive basis of $\mathbb{R}^3$. In this lecture, we call $\mathcal{F}$ the *Gauss frame* of f. In particular, 2-dimensional vector space spanned by $\{f_u(u,v), f_v(u,v)\}$ is the image of $T_{\mathrm{P}}U$ by the differential map:

$$\mathrm{Span}\{f_u(\mathrm{P}), f_v(\mathrm{P})\} = df(T_{\mathrm{P}}U).$$

We call this vector space by the *tangent vector space* of the surface at P. The tangent vector space is characterized as the orthogonal complement of $\nu(\mathrm{P})$, and we have the orthogonal decomposition

$$\mathbb{R}^3 = \left(T_{f(\mathrm{P})}\mathbb{R}^3\right) = df(T_{\mathrm{P}}U) \oplus \mathbb{R}\nu_{\mathrm{P}}. \tag{2.2}$$

From now on, we denote by

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2, \quad II = L\,du^2 + 2M\,du\,dv + N\,dv^2$$

the first and the second fundamental forms, respectively.

15. April, 2016.

The Weingarten Formula. The following formula mesures the change of the unit normal vector in terms of the entries of the fundamental forms (cf. Lemma 8.5 in [2-1] (page 85)):

Theorem 2.1 (The Weingarten Formula)**.** *It holds that*

$$\begin{cases} \nu_u = -A_1^1 f_u - A_1^2 f_v, \\ \nu_v = -A_2^1 f_u - A_2^2 f_v, \end{cases}$$
$$\left(A = \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}\right).$$

The Christoffel symbols and the Gauss Formula.

Definition 2.2 (The Christoffel symbols, [2-1], page 108)**.** The following Γ_{ij}^k $(i,j,k = 1,2)$ are called the *Christoffel symbols*:

$$\begin{cases} \Gamma_{11}^1 := \dfrac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, \\ \Gamma_{11}^2 := \dfrac{2EF_u - EE_v - FE_u}{2(EG - F^2)}, \\ \Gamma_{12}^1 = \Gamma_{21}^1 := \dfrac{GE_v - FG_u}{2(EG - F^2)}, \\ \Gamma_{12}^2 = \Gamma_{21}^2 := \dfrac{EG_u - FE_v}{2(EG - F^2)}, \\ \Gamma_{22}^1 := \dfrac{2GF_v - GG_u - FG_v}{2(EG - F^2)}, \\ \Gamma_{22}^2 := \dfrac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}. \end{cases} \tag{2.3}$$

By straightforward calculations, we have

Lemma 2.3.

$$
\begin{aligned}
E\Gamma^1_{11} + F\Gamma^2_{11} &= \frac{1}{2}E_u, & F\Gamma^1_{11} + G\Gamma^2_{11} &= F_u - \frac{1}{2}E_v \\
E\Gamma^1_{12} + F\Gamma^2_{12} &= \frac{1}{2}E_v, & F\Gamma^1_{12} + G\Gamma^2_{12} &= \frac{1}{2}G_u, \\
E\Gamma^1_{22} + F\Gamma^2_{22} &= F_v - \frac{1}{2}G_u & F\Gamma^1_{22} + G\Gamma^2_{22} &= \frac{1}{2}G_v, \\
\Gamma^1_{11} + \Gamma^2_{12} &= \frac{g_u}{2g}, & \Gamma^1_{21} + \Gamma^2_{22} &= \frac{g_v}{2g},
\end{aligned}
$$

where $g := EG - F^2$.

The Gauss formula ([2-1], Proposition 11.1 in page 122) is satated using the Christoffel symbols as follows:

Theorem 2.4 (The Gauss Formula)**.** *Let* $f\colon U \ni (u,v) \mapsto f(u,v) \in \mathbb{R}^3$ *be an immersion and* ν *its unit normal vector field, and let* Γ^i_{jk} $(i,j,k = 1,2)$ *and* L, M, N *be the Christoffel symbols and the entries of the second fundamental forms, respectively. Then it holds taht*

$$
\begin{cases}
f_{uu} = \Gamma^1_{11} f_u + \Gamma^2_{11} f_v + L\nu, \\
f_{uv} = \Gamma^1_{12} f_u + \Gamma^2_{12} f_v + M\nu, \\
f_{vv} = \Gamma^1_{12} f_u + \Gamma^2_{12} f_v + N\nu.
\end{cases}
\tag{2.4}
$$

The Gauss Frame and the Fundamental Equations. Combining Theorems 2.1 and 2.4, we have the following *Fundamental Equations for surface theory*:

Theorem 2.5. *Let* $f\colon U \ni (u,v) \mapsto f(u,v) \in \mathbb{R}^3$ *be an immersion of a domain* U *in the uv-plane. Then the Gauss frame* $\mathcal{F} := \{f_u, f_v, \nu\}$ *as in* (**??**) *satisfies the equations*

$$
\frac{\partial \mathcal{F}}{\partial u} = \mathcal{F}\Omega, \qquad \frac{\partial \mathcal{F}}{\partial v} = \mathcal{F}\Lambda
\tag{2.5}
$$
$$
\Omega := \begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} & -A^1_1 \\ \Gamma^2_{11} & \Gamma^2_{12} & -A^2_1 \\ L & M & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} \Gamma^1_{21} & \Gamma^1_{22} & -A^1_2 \\ \Gamma^2_{21} & \Gamma^2_{22} & -A^2_2 \\ M & N & 0 \end{pmatrix},
$$

where Γ^i_{jk} $(i,j,k = 1,2)$, A^k_l *and* L, M, N *are the Christoffel symbols, the entries of the Weingarten matrix and the entries of the second fundamental form, respectively.*

Since the coefficient matrices Ω, Λ in (2.5) are expressed in terms of the first and the second fundamenatal forms, we have the followoing, which is the "uniquness part" of the fundamental theorem for surface theory (which will be stated in Section **??**):

Corollary 2.6. *Let* f, $\tilde{f}\colon U \to \mathbb{R}^3$ *be two immersions of a domain* $U \subset \mathbb{R}^2$. *If the first and the second fundamental forms of* f *and* $\tilde{f}$ *are common, there exist a matrix* $P \in \mathrm{SO}(3)$ *and a vector* $\boldsymbol{p} \in \mathbb{R}^3$ *such that*

$$
\tilde{f} = Pf + \boldsymbol{p}.
\tag{2.6}
$$

In other words, the first and the second fundamental forms determines a surface uniquely up to (*orientation preserving*) *isometries of* $\mathbb{R}^3$.

Proof. Let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be the Gauss frames of f and $\tilde{f}$, respectivelymtively. By Theorem 2.5, they astisfy the equation (2.5) with common coefficient matrices Ω and Λ. Hence

$$\begin{aligned}\frac{\partial}{\partial u}\mathcal{F}\widetilde{\mathcal{F}}^{-1} &= \widetilde{\mathcal{F}}_u\mathcal{F}^{-1} + \widetilde{\mathcal{F}}\bigl(\mathcal{F}^{-1}\bigr)_u = \widetilde{\mathcal{F}}_u\mathcal{F}^{-1} - \widetilde{\mathcal{F}}\bigl(\mathcal{F}^{-1}\mathcal{F}_u\mathcal{F}^{-1}\bigr)\\ &= \widetilde{\mathcal{F}}\Omega\mathcal{F}^{-1} - \widetilde{\mathcal{F}}\Omega\widetilde{\mathcal{F}}^{-1} = O,\\ \frac{\partial}{\partial v}\widetilde{\mathcal{F}}\mathcal{F}^{-1} &= O.\end{aligned}$$

Since the domain D is connected, it follows that $\widetilde{\mathcal{F}}\mathcal{F}^{-1} = P$, for some constant matrix P, that is, $\widetilde{\mathcal{F}} = P\widetilde{\mathcal{F}}$ holds. Hence

$$\tilde{f}_u = Pf_u, \qquad \tilde{f}_v = Pf_v, \qquad \tilde{\nu} = P\nu$$

hold, where $\tilde{\nu}$ is the unit normal vector field of $\tilde{f}$. This implies

$$(\tilde{f} - Pf)_u = (\tilde{f} - Pf)_v = \mathbf{0},$$

and hence $\tilde{f} - Pf =: \boldsymbol{p}$ is a constant vector, which yields (2.6). So, it is sufficient to show that $P \in \mathrm{SO}(3)$. Fix P $:= (u_0, v_0) \in U$ and set

$$\boldsymbol{f}_1 = f_u(u_0, v_0), \qquad \boldsymbol{f}_2 = f_v(u_0, v_0), \qquad \boldsymbol{f}_3 = \nu(u_0, v_0).$$

Then

$$\boldsymbol{f}_1 \cdot \boldsymbol{f}_1 = E(u_0, v_0) = \tilde{f}_u(u_0, v_0) \cdot \tilde{f}_u(u_0, v_0) = P\boldsymbol{f}_1 \cdot P\boldsymbol{f}_1,$$

and similarly, we have

$$\boldsymbol{f}_j \cdot \boldsymbol{f}_k = P\boldsymbol{f}_j \cdot P\boldsymbol{f}_k \qquad (j, k = 1, 2, 3).$$

Since $\{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$ forms a basis of $\mathbb{R}^3$, we can conclude that

$$\boldsymbol{x} \cdot \boldsymbol{y} = P\boldsymbol{x} \cdot P\boldsymbol{y} \qquad (\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3),$$

and hence P is a orthogonal matrix. Moreover, both $\det \mathcal{F}$ and $\det \widetilde{\mathcal{F}}$ are positive since $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are positively oriented. Thus, $\det P > 0$, that is, $P \in \mathrm{SO}(3)$.

□

Theorema egregium Differentiate the first (resp. the second) equation of (2.4) in v and u, respectively, we obtain

$$\begin{aligned}f_{uuv} &= (*) + (\Gamma^2_{11,v} + \Gamma^1_{11}\Gamma^2_{11} + \Gamma^2_{11}\Gamma^2_{12} - LA^2_2)f_v,\\ f_{uvu} &= (*) + (\Gamma^2_{12,u} + \Gamma^1_{12}\Gamma^2_{12} + \Gamma^2_{12}\Gamma^2_{22} - MA^2_1)f_v,\end{aligned} \tag{2.7}$$

here $(*)$'s are linear cobinations of f_u and ν. Since $f_{uuv} = f_{uvu}$, we have

$$\begin{aligned}K = &\frac{E\bigl(E_vG_v - 2F_uG_v + {G_u}^2\bigr)}{4(EG - F^2)^2}\\ &+ \frac{F(E_uG_v - E_vG_u - 2E_vF_v - 2F_uG_u + 4F_uF_v)}{4(EG - F^2)^2}\\ &+ \frac{G\bigl(E_uG_u - 2E_uF_v + {E_v}^2\bigr)}{4(EG - F^2)^2} - \frac{E_{vv} - 2F_{uv} + G_{uu}}{2(EG - F^2)}\end{aligned} \tag{2.8}$$

comparing the coefficients of f_v in (2.7) and substituting (2.3), here K is the Gaussian curvature defined in (1.15). The equality (2.8) is known as Gauss' *Theorema Egregium* ("remarkable theorem"), cf. Theorem 11.2 of [2-1].

References

[2-1] 梅原雅顕・山田光太郎：曲線と曲面—微分幾何的アプローチ（改訂版），裳華房，2014.

[2-2] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, 1976.

Exercises

2-1H Assume $f\colon U \ni (u,v) \mapsto f(u,v) \in \mathbb{R}^3$ be an immersion of a domain U in $\mathbb{R}^2$, whose first and second fundamental forms are expressed as

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + dv^2, \qquad II = 2\sin\theta\, du\, dv,$$

where $\theta(u,v)$ is a smooth function in (u,v).

(1) Find the condition of θ for f to be an immersion.

(2) Compute the Christoffel symbols.

(3) Write down the equation (2.8) in terms of θ.