

3 The Gauss and Codazzi equations

The compatibility conditions. As seen in Theorem 2.5 in Section 2, the Gauss frame $\mathcal{F} = (f_u, f_v, \nu)$ for an immersion $f: D \ni (u, v) \mapsto f(u, v) \in \mathbb{R}^3$ satisfies the equation

$$(3.1) \quad \frac{\partial \mathcal{F}}{\partial u} = \mathcal{F}\Omega, \quad \frac{\partial \mathcal{F}}{\partial v} = \mathcal{F}\Lambda$$

$$\Omega := \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 & -A_1^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 & -A_1^2 \\ L & M & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} \Gamma_{21}^1 & \Gamma_{22}^1 & -A_2^1 \\ \Gamma_{21}^2 & \Gamma_{22}^2 & -A_2^2 \\ M & N & 0 \end{pmatrix},$$

where Γ_{jk}^i ($i, j, k = 1, 2$), A_l^k and L, M, N are the Christoffel symbols, the entries of the Weingarten matrix and the entries of the second fundamental form, respectively.

Lemma 3.1. *The coefficient matrices Ω, Λ in (3.1) satisfy*

$$(3.2) \quad \frac{\partial \Omega}{\partial v} - \frac{\partial \Lambda}{\partial u} = \Omega\Lambda - \Lambda\Omega.$$

Proof. Differentiating the first equation in (3.1) with respect to v , we have

$$\mathcal{F}_{uv} = (\mathcal{F}\Omega)_v = \mathcal{F}_v\Omega + \mathcal{F}\Omega_v = \mathcal{F}\Lambda\Omega + \mathcal{F}\Omega_v = \mathcal{F}(\Lambda\Omega + \Omega_v).$$

Similarly, differentiating the first equation in (3.1) in u , it holds that

$$\mathcal{F}_{vu} = \mathcal{F}(\Omega\Lambda + \Lambda_u).$$

Thus,

$$\mathcal{F}(\Lambda\Omega + \Omega_v) = \mathcal{F}(\Omega\Lambda + \Lambda_u)$$

holds. Noticing that \mathcal{F} is a regular matrix, we have the conclusion. \square

The equation (3.2) is called the *compatibility condition*, or the *integrability condition* of the equation (3.1).

The Gauss and Codazzi equations. The compatibility condition (3.2) consists of nine equations, because it is the equality for 3×3 matrices. However, they can be reduced three equations:

Lemma 3.2. *The compatibility condition (3.2) is equivalent to the equation (Equation (2.8))*

$$(3.3) \quad K = \frac{E(E_v G_v - 2F_u G_v + G_u^2)}{4(EG - F^2)^2} + \frac{F(E_u G_v - E_v G_u - 2E_v F_v - 2F_u G_u + 4F_u F_v)}{4(EG - F^2)^2} + \frac{G(E_u G_u - 2E_u F_v + E_v^2)}{4(EG - F^2)^2} - \frac{E_{vv} - 2F_{uv} + G_{uu}}{2(EG - F^2)}$$

and the following two equations:

$$(3.4) \quad \begin{aligned} L_v - M_u &= \Gamma_{21}^1 L + \Gamma_{21}^2 M - \Gamma_{11}^1 M - \Gamma_{11}^2 N, \\ M_v - N_u &= \Gamma_{22}^1 L + \Gamma_{22}^2 M - \Gamma_{12}^1 M - \Gamma_{12}^2 N. \end{aligned}$$

Proof. By a direct computations, we can conclude that the (1, 1), (1, 2), (2, 1), (2, 2)-components of (3.2) are equivalent to (2.8). On the other hand, the first (resp. the second) equation in (3.4) is equivalent to the (3, 1) (resp. (3, 2)) component of (3.2). Moreover, the (1, 3) and (2, 3)-components are equivalent to (3.4) because of the definition of the Weingarten matrix

$$A = \frac{1}{g} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \quad (g = EG - F^2)$$

and Lemma 2.3. \square

The equation (2.8) is called the *Gauss equation*. On the other hand, the equations (3.4) are called the *Codazzi equations*, or the *Codazzi-Mainardi equations*.

Corollary 3.3. *Let $f: \mathbb{R}^2 \supset D \rightarrow \mathbb{R}^3$ be an immersion with first and second fundamental forms as*

$$ds^2 = E du^2 + 2F du dv + G dv^2, \quad II = L du^2 + 2M du dv + N dv^2.$$

Then the entries E, F, G, L, M and N satisfy the Gauss equation (3.3) and the Codazzi equation (3.4), where Γ_{jk}^i 's are the Christoffel symbols (2.3), and K is the Gaussian curvature in (1.15).

References

- [3-1] 梅原雅頭・山田光太郎：曲線と曲面—微分幾何的アプローチ（改訂版），裳華房，2014.
- [3-2] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, 1976.

Exercises

3-1^H Assume that the first and second fundamental forms of the immersion $f: \mathbb{R}^2 \supset D \rightarrow \mathbb{R}^3$ are

$$ds^2 = E du^2 + 2F du dv + G dv^2, \quad II = 2M du dv.$$

(1) Show that the Codazzi equations are

$$M_u + \left(2\Gamma_{12}^2 - \frac{g_u}{2g} \right) M = 0, \quad M_v + \left(2\Gamma_{12}^1 - \frac{g_v}{2g} \right) M = 0,$$

where $g = EG - F^2$.

(2) Moreover, if the Gaussian curvature is **negative** constant, show that $E_v = 0$ and $G_u = 0$ hold.

3-2 Let f be an immersion of the uv -plane into \mathbb{R}^3 . The parameter (u, v) is said to be *isothermal* or *conformal* if the first fundamental form is written as

$$ds^2 = e^{2\sigma}(du^2 + dv^2) \quad (\text{i.e. } E = G = e^{2\sigma}, F = 0),$$

where $\sigma = \sigma(u, v)$ is a smooth function in (u, v) .²

Assume that f is parametrized by an isothermal parameter (u, v) .

(1) Show that the Gauss frame \mathcal{F} satisfies the equation

$$(3.5) \quad \frac{\partial \mathcal{F}}{\partial u} = \mathcal{F}\Omega, \quad \frac{\partial \mathcal{F}}{\partial v} = \mathcal{F}\Lambda$$

$$\Omega := \begin{pmatrix} \sigma_u & \sigma_v & -e^{-2\sigma}L \\ -\sigma_v & \sigma_u & -e^{-2\sigma}M \\ L & M & 0 \end{pmatrix},$$

$$\Lambda := \begin{pmatrix} \sigma_v & -\sigma_u & -e^{-2\sigma}M \\ \sigma_u & \sigma_v & -e^{-2\sigma}N \\ M & N & 0 \end{pmatrix},$$

where L , M and N are the entries of the second fundamental form.

²Let (Σ, ds^2) be an arbitrary 2-dimensional Riemannian manifold. Then, it is known that, for any point $P \in S$, there exists an isothermal coordinate chart (u, v) containing P , that is, the Riemannian metric ds^2 is written as $ds^2 = e^{2\sigma}(du^2 + dv^2)$ (cf. Section 15 of [3-1]).

(2) Verify that the Gauss and Codazzi equations are written as

$$\begin{aligned} \sigma_{uu} + \sigma_{vv} + e^{-2\sigma}(LN - M^2) &= 0 \\ L_v - M_u &= \sigma_v(L + N) \\ N_u - M_v &= \sigma_u(L + N). \end{aligned}$$

3-3 Let $f: \Sigma \rightarrow \mathbb{R}^3$ be an immersion of an oriented 2-manifold Σ , and (u, v) be an isothermal coordinate system around $P \in \Sigma$ compatible to the orientation of Σ , and (ξ, η) be another coordinate system around P compatible to the orientation of Σ .

(1) Show that (ξ, η) is isothermal if and only if

$$u_\xi = v_\eta, \quad u_\eta = -v_\xi.$$

(2) Verify that the above conditions are equivalent to that

$$\zeta := \xi + i\eta \mapsto z := u + iv$$

is holomorphic.³

³Hence, the existence of isothermal coordinates implies the existence of the structure of a Riemann surface (a 1-dimensional complex manifold) on an oriented Riemannian manifold.

3-4 Let $f: D \rightarrow \mathbb{R}^3$ be an immersion with an isothremal parameter (u, v) , with fundamental forms

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \quad II = L du^2 + 2M du dv + N dv^2.$$

(1) Show that the Gauss and Codazzi equations are equivalent to

$$-e^{-2\sigma}(\sigma_{uu} + \sigma_{vv}) = K, \quad \frac{1}{2} \frac{\partial H}{\partial z} = e^{-2\sigma} \frac{\partial q}{\partial \bar{z}},$$

where $z = u + iv$ be an complex coordinate,

$$q := \frac{1}{4}((L - N) - 2iM),$$

K is the Gaussian curvature, H is the mean curvature, and

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).$$

(2) When H is constant, verify that the Codazzi equation is equivalent to the holomorphicity of q .