4 The fundamental theorem for surfaces

We shall give a proof of the following theorem in this section (cf. Appendix B-10 in [4-1]):

Theorem 4.1 (The fundamental theorem for surface theory). *Let D be a simply connected domain of $\mathbb{R}^2$ and let $E(>0)$, F, $G(>0)$, L, M and N be a C^∞-functions on D satisfying $EG-F^2>0$, the Gauss equation (3.3), and the Codazzi equations (3.4). Then there exists an immersion $f\colon D\to\mathbb{R}^3$ whose first and second fundamental forms are*

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2,\quad II = L\,du^2 + 2M\,du\,dv + N\,dv^2.$$

Moreover, such an immersion f is unique up to rotations and parallel translations.

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Theorem 4.2 (The fundamental theorem). *Let V be a finite dimensional vector space over $\mathbb{R}$ and denote by $\operatorname{Hom}(V)$ the space of linear transformations on V. Take a C^∞-map $A\colon I\to\operatorname{Hom}V$ defined on an interval $I\subset\mathbb{R}$. Then for arbitrary t_0 and $\boldsymbol{v}_0\in V$, there exists a unique C^∞-map $\boldsymbol{v}\colon I\to V$ satisfying*

$$\text{(4.1)}\qquad \frac{d\boldsymbol{v}}{dt}(t) = A(t)\boldsymbol{v}(t),\qquad \boldsymbol{v}(t_0)=\boldsymbol{v}_0.$$

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The equation (4.1) is called an *initial value problem of a linear differential equation.*[4] We denote the unique solution of (4.1) by $\boldsymbol{v}_{A,t_0,\boldsymbol{v_0}}$.

Theorem 4.3. *Under the same notations as in Theorem 4.2, let $A\colon I\times U\to\operatorname{Hom}(V)$ and and $\boldsymbol{v}_0\colon I'\to V$ be C^∞-maps where I, I' are intervals and $U\subset\mathbb{R}^n$ is a domain. Then for arbitrarily fixed $t_0\in I$,*

$$\mathbb{R}^3\supset I\times U\times I'\ni(t,\boldsymbol{\alpha},\beta)\longmapsto \boldsymbol{v}_{A(*,\boldsymbol{\alpha}),t_0,\boldsymbol{v}_0(\beta)}\in V$$

is a C^∞-map.

Theorem 4.3 is called the *regularity of the solutions of ordinary differential equations with respect to parameters and initial conditions.*

From now on we denote by $\mathrm{M}(n,\mathbb{R})$ (resp. $\mathrm{GL}(n,\mathbb{R})$) the vector space consists of the $n\times n$-real matrices (resp. the $n\times n$-regular matrices).

Corollary 4.4. *Let $\Omega\colon I\to\mathrm{M}(n,\mathbb{R})$ be a C^∞-map defined on an interval I. Then for $t_0\in I$ and an arbitrary matrix $A_0\in\mathrm{M}(n,\mathbb{R})$, there exists a unique C^∞-map $\mathcal{F}_{A_0}\colon I\to\mathrm{M}(n,\mathbb{R})$ satisfying*

$$\text{(4.2)}\qquad \frac{d\mathcal{F}}{dt}(t)=\mathcal{F}(t)\Omega(t),\qquad \mathcal{F}(t_0)=A_0.$$

Moreover,

[4] Compare with the well-known Cauchy's existence theorem. The solution of the linear differential equation is defined on the whole interval I where the coefficient A is defined. See [4-2] and [4-3].

- *if* $A_0 \in \mathrm{GL}(n,\mathbb{R})$ *then* $\mathcal{F}(t) \in \mathrm{GL}(n,\mathbb{R})$, *for* $t \in I$,
- $\mathcal{F}_B = B\mathcal{F}_{\mathrm{id}}$, *where* id *is the* $n \times n$*-identity matrix and* $\mathcal{F}_B$ (*resp.* $\mathcal{F}_{\mathrm{id}}$) *is the solution of* (4.2) *with* $A_0 = B$ ($A_0 = \mathrm{id}$).

Proof. The first part is a direct conclusion of Theorem 4.2 for $V = \mathrm{M}(n,\mathbb{R})$ and $A(t)\colon V \ni F \mapsto \Omega(t)F \in V$.

Let $\mathcal{F}$ be the solution of (4.2). Then it holds that,

$$\frac{d}{dt}\det\mathcal{F} = \operatorname{tr}\left(\widetilde{\mathcal{F}}\frac{d\mathcal{F}}{dt}\right) = \operatorname{tr}(\widetilde{\mathcal{F}}\mathcal{F}\Omega) = \det\mathcal{F}\operatorname{tr}(\Omega),$$

where $\widetilde{\mathcal{F}}$ is the cofactor matrix of $\mathcal{F}$. Then $f := \det\mathcal{F}$ satisfies

$$\frac{df}{dt} = f\omega, \quad f(t_0) = a_0, \quad \text{where } \omega = \operatorname{tr}\Omega \text{ and } a_0 = \det A_0.$$

The unique solution of above equation is

$$f(t) = a_0 \exp\left(\int_{t_0}^{t} \omega(s)\,ds\right),$$

which never vanish if $a_0 \neq 0$. Final assertion holds by the uniqueness of the solution of (4.2). □

Integrable Partial Differential Equations. Let D be a domain in the uv-plane $\mathbb{R}^2$ and take C^∞ maps $\Omega, \Lambda\colon D \to \mathrm{M}(n,\mathbb{R})$. In this section we consider a system of differential equations of unknown $\mathcal{F}\colon D \to \mathrm{M}(n,\mathbb{R})$:

$$\frac{\partial\mathcal{F}}{\partial u} = \mathcal{F}\Omega, \qquad \frac{\partial\mathcal{F}}{\partial v} = \mathcal{F}\Lambda, \quad \mathcal{F}(\mathrm{P}) = F_0 \in \mathrm{GL}(n,\mathbb{R}), \tag{4.3}$$

where $\mathrm{P} \in D$ is a fixed point.

Lemma 4.5. *Assume that there exists a solution* $\mathcal{F}$ *of* (4.3). *Then* $\mathcal{F}(u,v) \in \mathrm{GL}(n,\mathbb{R})$ *for any* $(u,v) \in D$ *and it holds that*

$$\Omega_v - \Lambda_u = \Omega\Lambda - \Lambda\Omega. \tag{4.4}$$

Proof. Fix $\mathrm{Q} \in D$ and take a smooth path $\gamma(t) = \bigl(u(t), v(t)\bigr)$ $(0 \leqq t \leqq 1)$ on D joining P and Q. Then $\mathcal{F}\circ\gamma(t)\colon [0,1] \to \mathrm{M}(n,\mathbb{R})$ satisfies

$$\frac{d\mathcal{F}\circ\gamma}{dt}(t) = \mathcal{F}\circ\gamma(t)\hat{\Omega}(t), \quad \mathcal{F}\circ\gamma(0) = F_0 \in \mathrm{GL}(n,\mathbb{R}), \qquad \hat{\Omega}(t) := \Omega\circ\gamma(t)\dot{u}(t) + \Lambda\circ\gamma(t)\dot{v}(t). \tag{4.5}$$

Then Corollary 4.4 implies that $\mathcal{F}(\mathrm{Q}) \in \mathrm{GL}(n,\mathbb{R})$. Since Q is arbitrary, the first assertion holds.

The second assertion can be proven by the same way in the proof of Lemma 3.1. □

Theorem 4.6. *Let* D *be a* simply connected *domain in* $\mathbb{R}^2$. *Then there exists a unique solution* $\mathcal{F}\colon D \to \mathrm{M}(n,\mathbb{R})$ *of* (4.3) *if* Ω *and* Λ *satisfy* (4.4).

Proof. First we shall prove the uniqueness: Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the solutions of (4.3). Since the values of $\mathcal{F}_j$ are regular matrices (Lemma 4.5), we can set $\mathcal{G} := \mathcal{F}_1\mathcal{F}_2^{-1}$. Then by the similar computation in the proof of Corollary 2.6, we have $\mathcal{G}_u = \mathcal{G}_v = O$, and hence $\mathcal{G}$ is constant on D:

$$\mathcal{G}(\mathrm{P}) = \mathcal{F}_1(\mathrm{P})\mathcal{F}_2(\mathrm{P})^{-1} = F_0F_0^{-1} = \mathrm{id}.$$

Then we have $\mathcal{F}_1 = \mathcal{F}_2$.

Next, we prove the existence. Take $\mathrm{Q} \in D$ arbitrarily and choose a path $\gamma(t) = \bigl(u(t), v(t)\bigr)$ $(0 \leqq t \leqq 1)$ joining P and Q, and consider the ordinary differential equation (4.5). Let $\mathcal{F}_\gamma \colon I \to \mathrm{GL}(n, \mathbb{R})$ be the unique solution (cf. Corollary 4.4) of (4.5), and set $\mathcal{F}(\gamma, \mathrm{Q}) := \mathcal{F}_\gamma(1)$.

We now prove that $\mathcal{F}$ does not depend on the choice of the path γ. Take another path $\tilde{\gamma}$ joining P and Q. Since D is simply connected, they are homotopically equivalent. In other words, we can take a smooth map $\sigma \colon [0,1] \times [0,1] \to D$ such that $\sigma(0,t) = \gamma(t)$, $\sigma(1,t) = \tilde{\gamma}(t)$, $\sigma(s,0) = \mathrm{P}$, $\sigma(s,1) = \mathrm{Q}$. We write $\sigma(s,t) = \bigl(u(s,t), v(s,t)\bigr)$ and set

$$S = \Omega \circ \sigma u_s + \Lambda \circ \sigma v_s, \quad T = \Omega \circ \sigma u_t + \Lambda \circ \sigma v_t.$$

Note that

$$(4.6) \qquad S(s,1) = O \qquad (0 \leqq s \leqq 1),$$

because $\sigma(s,1)$ is constant. For each fixed $s \in [0,1]$, take the unique solution $\hat{\mathcal{F}}(s,t)$ of the ordinary differential equation

$$(4.7) \qquad \frac{\partial \hat{\mathcal{F}}(s,t)}{\partial t} = \hat{\mathcal{F}}(s,t) T(s,t), \qquad \hat{\mathcal{F}}(s,0) = F_0.$$

Then by the regularity of the solution of ordinary differential equation with respect to the parameters, we have a smooth map $\hat{\mathcal{F}} \colon [0,1] \times [0,1] \to D$, and by definition,

$$F_0 = \hat{\mathcal{F}}(s,0), \quad \mathcal{F}(\gamma, \mathrm{Q}) = \hat{\mathcal{F}}(0,1), \quad \mathcal{F}(\tilde{\gamma}, \mathrm{Q}) = \hat{\mathcal{F}}(1,1),$$

that is, to show that $\mathcal{F}(\gamma, \mathrm{Q})$ does not depend on γ, it is sufficient to show that $\hat{\mathcal{F}}(0,1) = \hat{\mathcal{F}}(1,1)$. Noticing $S_t - T_s - ST + TS = O$

holds because of (4.4), we have

$$\begin{aligned}\bigl(\hat{\mathcal{F}}_s - \hat{\mathcal{F}} S\bigr)_t &= \hat{\mathcal{F}}_{st} - \hat{\mathcal{F}}_t S - \hat{\mathcal{F}} S_t \\ &= \hat{\mathcal{F}}_{ts} - \mathcal{F} T S - \mathcal{F} S_t = (\hat{\mathcal{F}}_s - \hat{\mathcal{F}} S) T.\end{aligned}$$

Hence for each fixed s, $\hat{\mathcal{F}}_s - \hat{F} S$ is another solution of the same equation (4.7) with the initial condition $\hat{\mathcal{F}}_s(s,0) - \hat{\mathcal{F}}(s,0) S(s,0) = O$. Hence $\hat{\mathcal{F}}_s - \hat{\mathcal{F}} S = O$ for $(s,t) \in [0,1] \times [0,1]$. In particular, $\hat{\mathcal{F}}_s(s,1) = \hat{\mathcal{F}}(s,1) S(s,1) = O$ and then $\hat{\mathcal{F}}(s,1)$ is constant.

Thus, by setting $\mathcal{F}(\mathrm{Q}) := \mathcal{F}(\gamma, \mathrm{Q})$, we have the map $\mathcal{F} \colon D \to \mathrm{M}(n, \mathbb{R})$. We finally prove that $\mathcal{F}$ satisfies the equation (4.3). Let $\mathrm{Q} = (u_0, v_0)$, $\mathrm{Q}_h = (u_0 + h, v_0)$ and set $\gamma(t) = (u_0 + th, v_0)$ $(t \in [0,1])$. Then $\mathcal{F}(\mathrm{Q}_h) = \hat{\mathcal{F}}(1)$, where $\hat{\mathcal{F}}$ is a solution of

$$\frac{d\hat{\mathcal{F}}}{dt} = h \hat{\mathcal{F}} \Omega \circ \gamma(t), \qquad \hat{\mathcal{F}}(0) = \mathcal{F}(\mathrm{Q}).$$

Thus, we can show

$$\mathcal{F}_u(\mathrm{Q}) = \lim_{h \to 0} \frac{\mathcal{F}(\mathrm{Q}_{-h}) - \mathcal{F}(\mathrm{Q})}{h} = \mathcal{F}(\mathrm{Q}) \Omega(\mathrm{Q}).$$

Similarly, we have $\mathcal{F}_v = \mathcal{F} \Lambda$. □

Corollary 4.7 (Poincaré Lemma)**.** *Let $\alpha := \omega\, du + \lambda\, dv$ be a differential one form on a simply connected domain $D \subset \mathbb{R}^2$. If $d\alpha = (\lambda_u - \omega_v) du \wedge dv = 0$, there exists a smooth function $f \colon D \to \mathbb{R}$ such that $df = \alpha$.*

Proof. Consider the equation $\varphi_u = \varphi\omega$, $\varphi_v = \varphi\lambda$ and apply Theorem 4.6 for $n = 1$. Letting $f = e^\varphi$, we have the desired function. □

Proof of Theorem 4.1. The uniqueness is already shown in Corollary 2.6. We show the existence. Consider the equation (3.1). with initial condition at $\mathrm{P} \in D$

$$\mathcal{F}(\mathrm{P}) := \begin{pmatrix} \sqrt{E_0} & F_0/\sqrt{E_0} & 0 \\ 0 & \sqrt{(E_0G_0 - F_0^2)/E_0} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $E_0 = E(\mathrm{P})$, ... Then by Theorem 4.6, there exists the unique solution $\mathcal{F}\colon D \to \mathrm{GL}(3,\mathbb{R})$. Write $\mathcal{F} = (\omega, \lambda, \nu)$. Then by the equation (3.1), $\omega_v = \lambda_u$, that is, $\mathbb{R}^3$-valued one form $\alpha = \omega\, du + \lambda\, dv$ is closed. Then by the Poincaré lemma (Corollary 4.7), there exists a smooth map $f\colon D \to \mathbb{R}^3$ such that $f_u = \omega$, $f_v = \lambda$. We show that f is the desired surface. Let

$$\mathcal{H} := {}^t\mathcal{F}\mathcal{F} = \begin{pmatrix} f_u \cdot f_u & f_u \cdot f_v & f_u \cdot \nu \\ f_v \cdot f_u & f_v \cdot f_v & f_v \cdot \nu \\ \nu \cdot f_u & \nu \cdot f_v & \nu \cdot \nu \end{pmatrix}, \qquad \bigl(\mathcal{F} = (f_u, f_v, \nu)\bigr).$$

Take an arbitrary $\mathrm{Q} \in D$ and a path γ joining P and Q. Then $\hat{\mathcal{H}} = \mathcal{H} \circ \gamma$ satisfies the linear ordinary equation

$$\text{(4.8)} \qquad \frac{d\hat{\mathcal{H}}}{dt} = {}^t\hat{\Omega}\hat{\mathcal{H}} + \hat{\mathcal{H}}\hat{\Omega}$$

where $\hat{\Omega}(t)$ is as in (4.5). On the other hand,

$$\hat{\mathcal{H}}_0 = \mathcal{H}_0 \circ \gamma, \qquad \mathcal{H}_0 = \begin{pmatrix} E & F & 0 \\ F & G & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a solution of (4.8) with same initial condition as $\hat{\mathcal{H}}$ (cf. Problem 4-1). Thus $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0$ by the uniqueness part of Theorem 4.2. Since Q is arbitrary, we have

$$f_u \cdot f_u = E,\ f_u \cdot f_v = F,\ f_v \cdot f_v = G,\ f_u \cdot \nu = f_v \cdot \nu = 0,\ |\nu| = 1.$$

Hence the entries of first fundamental form of f is E, F, G and ν is the unit normal vector. Then by (3.1), we can show that the entries of the second fundamental form are L, M and N.

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Exercises

4-1[H] Let Ω and Λ be as in (3.1). Prove that

$$\mathcal{H} := \begin{pmatrix} E & F & 0 \\ F & G & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

satisfies the equation

$$\frac{\partial \mathcal{H}}{\partial u} = {}^t\Omega\mathcal{H} + \mathcal{H}\Omega, \qquad \frac{\partial \mathcal{H}}{\partial v} = {}^t\Lambda\mathcal{H} + \mathcal{H}\Lambda.$$