5 The Asymptotic Chebyshev Nets

Asymptotic directions. Let $f\colon \mathbb{R}^2 \supset D \to \mathbb{R}^3$ be an immersion and fix $\mathrm{P} = (u_0, v_0) \in D$. Consider a curve $\gamma(t) = f\bigl(u(t), v(t)\bigr)$ with $\gamma(0) = f(\mathrm{P})$. We define the *normal curvature* of $\gamma(t)$ at P as

$$\kappa_n(\gamma, \mathrm{P}) := \left(\frac{\ddot\gamma(0)}{|\dot\gamma(0)|^2}\right) \cdot \nu(\mathrm{P}), \tag{5.1}$$

where ν is the unit normal vector field of f.

Under the situations above, we have

$$\kappa_n(\gamma, \mathrm{P}) := \frac{L\,\dot u^2 + 2M\,\dot u\dot v + N\,\dot v^2}{E\,\dot u^2 + 2F\,\dot u\dot v + G\,\dot v^2}, \tag{5.2}$$

where E, F, G, L, M, and N are the entry of the first and second fundamental forms, which are evaluated at P, and $(\dot u, \dot v) = \bigl(\dot u(0), \dot v(0)\bigr)$.

In fact, by the chain rule, we have

$$\begin{aligned} \dot\gamma(0) &= \left.\frac{d}{dt}\right|_{t=0} f\bigl(u(t), v(t)\bigr) = f_u\dot u + f_v\dot v, \\ \ddot\gamma(0) &= f_u\ddot u + f_v\ddot v + f_{uu}\dot u^2 + 2f_{uv}\dot u\dot v + f_{vv}\dot v^2, \end{aligned}$$

where $\dot u$, $\ddot u$ etc. are evaluated at $t = 0$, and f_u, f_{uu} etc. are evaluated at P. Since f_u and f_v are perpendicular to ν and $L = f_{uu} \cdot \nu$, etc, we have (5.2). By (5.2), the normal curvature at P depends only on the velocity vector $\boldsymbol{v} = \dot\gamma(0)$ of $\gamma(t)$ at P. Moreover, it depends only on the direction of $\boldsymbol{v}$. So we write

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$$\kappa_n(\boldsymbol{v}) := \kappa_n(\gamma, \mathrm{P}), \qquad \boldsymbol{v} = \dot\gamma(0). \tag{5.3}$$

Theorem 5.1 (Proposition 9.5 in [5-1])**.** *The maximum and minimum of the normal curvature at* P *are the principal curvatures.*

Proof. Since $\kappa_n(\boldsymbol{v})$ depends only the direction of $\boldsymbol{v}$, then it can be considered as a function defined on S^1. Then it has the maximum and minimum. By (5.2), the maximum and minimum of κ_n are the maximum and minimum of

$$h(\alpha, \beta) := L\alpha^2 + 2M\alpha\beta + N\beta^2 \qquad \text{under the condition} \\ g(\alpha, \beta) := E\alpha^2 + 2F\alpha\beta + G\beta^2 = 1.$$

Let λ be the Lagrange multiplier. Then if κ_n takes maximum or minimum at (α, β) $(\neq (0, 0))$, $(h - \lambda g)_\alpha = (h - \lambda g)_\beta = 0$:

$$(L - \lambda E)\alpha + (M - \lambda F)\beta = 0, \qquad (M - \lambda F)\alpha + (N - \lambda G)\beta = 0.$$

This system admit a solution $(\alpha, \beta) \neq (0, 0)$ if and only if

$$\det\begin{pmatrix} L - \lambda E & M - \lambda F \\ M - \lambda F & N - \lambda G \end{pmatrix} = 0 \tag{5.4}$$

and in this case, $\lambda = \kappa_n$ is the maximum or minimum of $\kappa_n(\boldsymbol{v})$. Since (5.4) holds if and only if

$$\det\left[\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] = 0,$$

that is, λ is an eigenvalue of A as in (1.13). Hence the maximum and minimum of κ_n are the principal curvatures. □

Corollary 5.2. *If the Gaussian curvature K is negative at P, there exists two linearly independent directions $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ of the tangent space at P such that $\kappa_n(\boldsymbol{v}_j) = 0$.*

Proof. Since $K(\mathrm{P}) < 0$, the principal curvatures λ_1 and λ_2, the maximum and the minimum of $\kappa_n(\boldsymbol{v})$, have opposite signs. □

Definition 5.3. The directions $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ as in Corollary 5.2 is called the *asymptotic directions*.

Fact 5.4 (Theorem 9.9, Figure 8.1 in [5-1])**.** *At a point P with $K(\mathrm{P}) < 0$, the intersection of the surface and the tangent plane of the surface at P consists of two curves intersecting at P, and the tangent directions of these curves are the asymptotic directions.*

Fact 5.5 (Theorem B-5.4 in [5-1])**.** *Let P be a point on the surface with $K(\mathrm{P}) < 0$. Then there exists a local parameter (u, v) on a neighborhood U of P such that the u-direction and v-direction are the asymptotic directions on each point U.*

Definition 5.6. The coordinate system as in Fact 5.5 is called the *asymptotic coordinate system*.

Proposition 5.7. *A parameter (u, v) of the surface is asymptotic coordinate system if and only if the second fundamental form is in the form*

$$II = 2M\,du\,dv,$$

that is, $L = N = 0$.

Proof. Let $\mathrm{P} = (u_0, v_0)$. Then the normal curvature of the u-direction (resp. the v-direction) is $(f_{uu}/|f_u|^2)\cdot\nu = L/E$ (resp. $(f_{vv}/|f_v|^2)\cdot\nu = N/G$). The coordinate system (u, v) is asymptotic if and only if these two normal curvatures vanish, that is, $L = N = 0$. □

Example 5.8. Consider a parabolic hyperboloid $z = \frac{1}{2}(x^2 - y^2)$. Since this surface is parametrized as $(x, y) \mapsto \left(x, y, \frac{1}{2}(x^2 - y^2)\right)$, the first and second fundamental forms are

$$ds^2 = (1+x^2)\,dx^2 - 2xy\,dx\,dy + (1+y^2)\,dy^2, \quad II = \frac{dx^2 - dy^2}{\sqrt{1 + x^2 + y^2}}.$$

Since $dx^2 - dy^2 = (dx + dy)(dx - dy) = d(x + y)d(x - y)$, the parameter change $u = x + y$, $v = x - y$ yields

$$II = \frac{du\,dv}{\sqrt{1 + \frac{1}{2}u^2 + \frac{1}{2}v^2}}.$$

Hence (u, v) is the asymptotic coordinate system. The surface is represented by

$$(u, v) \longmapsto \left(\frac{u+v}{2}, \frac{u-v}{2}, \frac{uv}{2}\right).$$

Asymptotic Chebyshev net.

Theorem 5.9. *Let $f\colon \Sigma \to \mathbb{R}^3$ be an immersion of 2-dimensional manifold Σ into the Euclidean 3-space, whose Gaussian curvature is -1. Then for each point P, there exists a local coordinate* *system (u, v) on a neighborhood of* P *such that the first and second fundamental forms are represented by*

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + dv^2, \qquad II = 2\sin\theta\, du\, dv. \tag{5.5}$$

Here, θ is a smooth function in (u, v) satisfying

$$\theta_{uv} = \sin\theta. \tag{5.6}$$

The coordinate system (u, v) in Theorem 5.9 is called the *asymptotic Chebyshev net* and (5.6) is called the *sine Gordon equation.* Here function θ in (5.5) is the angle between the two asymptotic directions.

Proof. Let (u, v) be an asymptotic coordinate system around P (cf. Fact 5.5). Then the first and second fundamental forms are in the form

$$ds^2 = E\, du^2 + 2F\, du\, dv + G\, dv^2, \qquad II = 2M\, du\, dv.$$

Then by Problem 3-1, the Codazzi equations yield

$$E_v = 0, \qquad G_u = 0.$$

Hence E and G depends only on u and v, respectively:

$$E = E(u), \qquad G = G(v).$$

Since E and G are positive, there exists a function $\xi = \xi(u)$, $\eta = \eta(v)$ such that

$$\xi_u = \sqrt{E(u)}, \qquad \eta_v = \sqrt{G(v)}.$$

Then (ξ, η) is the desired coordinate system. Then the fundamental forms are

$$ds^2 = d\xi^2 + 2\,\widetilde{F}\, d\xi\, d\eta + d\eta^2, \qquad II = 2\widetilde{M}\, d\xi\, d\eta.$$

Since the Gaussian curvature is -1, that is,

$$K = \frac{-\widetilde{M}^2}{1 - \widetilde{F}^2} = -1,$$

we have

$$\widetilde{M}^2 + \widetilde{F}^2 = 1.$$

So there exists a smooth function θ such that $\widetilde{M} = \sin\theta$ and $\widetilde{F} = \cos\theta$. Thus we have the desired coordinate system. Moreover, by Problem 2-1, θ satisfies $\theta_{\xi\eta} = \sin\theta$ (which is equivalent to the Gauss equation). □

Remark 5.10. The asymptotic Chebyshev net is unique up to the coordinate changes

$$(u, v) \mapsto (\pm u + a, \pm v + b), \qquad (u, v) \mapsto (v, u).$$

References

[5-1] 梅原雅顕・山田光太郎：曲線と曲面—微分幾何的アプローチ（改訂版），裳華房，2014.

Exercises

5-1 Consider a smooth map $f\colon D \to \mathbb{R}^3$ as (cf. Problem 1-1)

$$f(u,v) = \left(\frac{\cos u}{\cosh v}, \frac{\sin u}{\cosh v}, v - \tanh v\right),$$

where $D = \{(u,v) \mid v > 0\}$.

(1) Write down the first fundamental and second fundamental forms in terms of (u,v).

(2) Find parameter change $(u,v) \mapsto (\xi,\eta)$ to the asymptotic Chebyshev net (ξ,η).

(3) Find the asymptotic angle $\theta(\xi,\eta)$.