6 Surfaces of constant negative curvature—the sine Gordon equation

Surfaces of constant negative curvature. As a corollary to Theorem 5.9 (the existence of asymptotic Chebyshev net) and the fundamental theorem for surface theory (Theorem 4.1), we have

Theorem 6.1. *For a function $\theta = \theta(u, v)$ defined on a simply connected region D on $\mathbb{R}^2$ satisfying $\theta_{uv} = \sin\theta$ and*

$$\theta(u, v) \in (0, \pi) \qquad \bigl((u, v) \in D\bigr) \tag{6.1}$$

there exists a unique immersion $f\colon D \to \mathbb{R}^3$ (up to congruence of $\mathbb{R}^3$) with first and second fundamental forms as

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + dv^2, \qquad II = 2\sin\theta\, du\, dv. \tag{6.2}$$

Conversely, any surfaces in $\mathbb{R}^3$ with constant curvature -1 is obtained in this way.

As mentioned in Section 5, the equation

$$\theta_{uv} = \sin\theta. \tag{6.3}$$

Theorem 6.1 claims that the solutions of the sine-Gordon equation with

Example 6.2. Let

$$\theta(u, v) = 4\tan^{-1}\exp(u + v). \tag{6.4}$$

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Then one can easily see that it satisfies the sine-Gordon equation, and satisfies (6.1) on a domain $D = \{(u, v) \mid u + v < 0\}$.

If we set $\xi := u - v$, $\eta := u + v$, the first and second fundamental forms can be written as

$$ds^2 = \frac{1}{\cosh^2\xi}(d\xi^2 + \sinh^2\eta\, d\eta^2), \quad II = \frac{\tanh\eta}{\cosh\eta}(-d\xi^2 + d\eta^2),$$

which coincide with the fundamental forms of the pseudosphere (Problem 1-1):

$$f(\xi, \eta) = \left(\frac{\cos\xi}{\cosh\eta}, \frac{\sin\xi}{\cosh\eta}, \eta - \tanh\eta\right).$$

The third fundamental form and the flat structure. Let $f\colon D \to \mathbb{R}^3$ be an immersion and $\nu\colon D \to S^2 \subset \mathbb{R}^3$ its unit normal vector field, where S^2 is considered as the set of unit vectors of $\mathbb{R}^3$.

Definition 6.3. The *third fundamental* form of f is the metric on D induced by the map ν:

$$III := d\nu \cdot d\nu := (\nu_u \cdot \nu_u)\, du^2 + 2(\nu_u \cdot \nu_v)\, du\, dv + (\nu_v \cdot \nu_v)\, dv^2,$$

where (u, v) is a local coordinate system on D.

Lemma 6.4. *The third fundamental form satisfies*

$$III - 2H\,II + K\, ds^2 = 0,$$

where H and K are the mean and the Gauss curvatures of f, and ds^2 and II are the first fundamental forms, respectively.

Proof. Fix a local coordinate system (u, v) and let $\widehat{I}$ and $\widehat{II}$ be the first and second fundamental matrices, respectively. Then the Weingarten matrix A is defined as $A := \widehat{I}^{-1}\widehat{II}$. Here, by the Weingarten formula (Theorem 2.1), it holds that

$$(\nu_u, \nu_v) = -(f_u, f_v)A.$$

Then the matrix representation (the third fundamental matrix) of $\widehat{III}$ is computed as

$$\begin{aligned}\widehat{III} &= \begin{pmatrix} {}^t\nu_u \\ {}^t\nu_v \end{pmatrix}(\nu_u, \nu_v) = {}^tA \begin{pmatrix} {}^tf_u \\ {}^tf_v \end{pmatrix}(f_u, f_v)A \\ &= {}^t\widehat{II}\,{}^t\widehat{I}^{-1}\,\widehat{I}\,\widehat{I}^{-1}\,\widehat{II} = \widehat{II}\,\widehat{I}^{-1}\,\widehat{II} = \widehat{I}\left(\widehat{I}^{-1}\,\widehat{II}\right)^2 = \widehat{I}A^2.\end{aligned}$$

On the other hand, by the Cayley-Hamilton formula we have

$$A^2 - (\operatorname{tr} A)A + (\det A)I = A^2 - 2HA + KI = O,$$

where I and O are the 2×2 identity matrix and the zero matrix, respectively. Thus, we have

$$O = \widehat{I}A^2 - 2H\,\widehat{I}A + K\,\widehat{I}\ \widehat{III} - 2H\,\widehat{II} + K\,\widehat{I},$$

and hence we have the conclusion. □

Theorem 6.5. *Let $f\colon D \to \mathbb{R}^3$ be an immersion with constant Gaussian curvature -1, and let ν be its unit normal vector field. Then $ds^2 + III$ is a flat metric, that is, a Riemann metric of constant Gaussian curvature 0.*

Proof. Take the asymptotic Chebyshev net (u, v) as

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + dv^2, \quad II = 2\sin\theta\, du\, dv.$$

Then the Weingarten matrix is expressed as

$$A = \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & \sin\theta \\ \sin\theta & 0 \end{pmatrix} = \begin{pmatrix} -\cot t & \csc t \\ \csc t & -\cot t \end{pmatrix},$$

and thus the mean curvature H is $-\cot t$. Thus, by Lemma 6.4,

$$\widehat{III} = -2\cot t\,\widehat{II} + \widehat{I} = \begin{pmatrix} 1 & -\cos\theta \\ -\cos\theta & 1 \end{pmatrix}.$$

Hence

$$\widehat{I} + \widehat{III} = 2I,$$

that is, $ds^2 + III = 2(du^2 + dv^2)$ which is a flat metric. □

Remark 6.6. It is known that a complete, simply connected flat (with zero Gaussian curvature) Riemannian manifold (M, ds^2) is isometric to $\mathbb{R}^2$ with the canonical metric. We consider a complete immersion $f\colon M \to \mathbb{R}^3$ with constant Gaussian curvature. Since the induced metric ds^2 is complete, so is $d\sigma^2 := ds^2 + III$. Then the universal cover $(\widetilde{M}, d\tilde{\sigma}^2)$ of $(M, d\sigma^2)$ is isometric to the Euclidean plane.

Equations for the orthonormal frame. Let $f\colon D \to \mathbb{R}^3$ be a surface of constant Gaussian curvature -1 with unit normal

vector field ν, and (u,v) the asymptotic Chebyshev net with (6.2), We set
(6.5)
$$\boldsymbol{e}_1 := \frac{1}{2}\sec\frac{\theta}{2}(f_u+f_v), \quad \boldsymbol{e}_2 := \frac{1}{2}\csc\frac{\theta}{2}(-f_u+f_v), \quad \boldsymbol{e}_3 := \nu.$$
Then one can easily see that

$$\mathcal{G} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) \tag{6.6}$$

is an orthogonal matrix for each (u,v). We call $\mathcal{G}$ the *orthonormal frame associated to the Chebyshev net* (u,v).

Lemma 6.7. *The orthonormal frame* (6.6) *satisfies*

$$\frac{\partial\mathcal{G}}{\partial u} = \mathcal{G}U, \quad \frac{\partial\mathcal{G}}{\partial v} = \mathcal{G}V, \tag{6.7}$$
$$U = \frac{1}{2}\begin{pmatrix} 0 & \theta_u & \sin\frac{\theta}{2} \\ -\theta_u & 0 & \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & -\cos\frac{\theta}{2} & 0 \end{pmatrix},$$
$$V = \frac{1}{2}\begin{pmatrix} 0 & -\theta_u & \sin\frac{\theta}{2} \\ \theta_v & 0 & -\cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} & 0 \end{pmatrix}.$$

Proof. Direct computations from (6.5) and Theorem 2.5. Moreover, the integrability condition $U_v - V_u = UV - VU$ (cf. (4.4)) is equivalent to the sine-Gordon equation $\theta_{uv} = \sin\theta$. □

Extension of constant negative curvature surfaces. The advantage of (6.7) is that it is valid even if $\theta \equiv 0 \pmod{\pi}$. Thus, we have

Theorem 6.8. *Let $\theta\colon D \to \mathbb{R}^3$ be a smooth function on an simply connected domain D in the uv-plane satisfying the sine-Gordon equation* (6.3). *Then their exists a smooth map $f\colon D \to \mathbb{R}^3$ and $\nu\colon D \to S^2 \subset \mathbb{R}^3$ such that*

$$f_u\cdot\nu = 0, \quad f_v\cdot\nu = 0, \quad (\nu\cdot\nu = 1), \tag{6.8}$$

and

$$\begin{aligned} ds^2 &:= df\cdot df = du^2 + 2\cos\theta\,du\,dv + dv^2, \\ I\!I &:= -d\nu\cdot df = 2\sin\theta\,du\,dv. \end{aligned} \tag{6.9}$$

Moreover, f is an immersion of constant Gaussian curvature -1 on the regions $\{(u,v)\,|\,\theta(u,v)\not\equiv 0 \pmod{\pi}\}$.

Proof. Since sine-Gordon equation is the integrability condition for (6.7). So there exists a solution $\mathcal{G}$ with the initial condition $\mathcal{G}(\mathrm{P}_0) = I$, where I is the identity matrix. Since both U and V are skew symmetric matrices, $\mathcal{G}$ takes its values the set of orthogonal matrices. In fact, one can easily show

$$({\mathcal{G}}^t\mathcal{G})_u = ({\mathcal{G}}^t\mathcal{G})_v = O.$$

Let $\mathcal{G} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$. Then by the equation (6.7), the $\mathbb{R}^3$-valued 1-form

$$\omega := \left(\cos\frac{\theta}{2}\boldsymbol{e}_1 - \sin\frac{\theta}{2}\boldsymbol{e}_2\right)du + \left(\cos\frac{\theta}{2}\boldsymbol{e}_1 + \sin\frac{\theta}{2}\boldsymbol{e}_2\right)dv$$

is closed, that is, $d\omega = 0$. Then by the Poincaré Lemma (Corollary 4.7), there exists $f\colon D \to \mathbb{R}^3$ with $df = \omega$. This f is the desired one. □

Remark 6.9. Though the map $f\colon D \to \mathbb{R}^3$ has singular points on the set $\Sigma := \{(u,v) \in D \,|\, \theta(u,v) \equiv 0 \pmod{\pi}\}$, the unit normal vector field $\nu = \boldsymbol{e}_3$ is defined on Σ. A map $f\colon D \to \mathbb{R}^3$ is said to be a *frontal* if there exists a unit normal vector field $\nu\colon D \to S^2$, that is, ν satisfies (6.8). Moreover, if a smooth map $(f,\nu)\colon D \to \mathbb{R}^3 \times S^2$ is an immersion, f is called a *front* of a *wave front*. Various differential geometric properties for wave fronts are treated in [6-3], and will be treated in [6-2].

In these terms, our f in Theorem 6.8 is a front, because $ds^2 + I\!I\!I = 2(du^2 + dv^2)$ is positive definite, that is, (f,ν) is an immersion.

Example 6.10. The constant function $\theta(u,v) = 0$ satisfies the sine-Gordon equation (6.3). Then

$$\mathcal{G} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(u-v) & -\sin(u-v) \\ 0 & \sin(u-v) & \cos(u-v) \end{pmatrix}$$

is the solution of (6.7) with $\mathcal{G}(0,0) = I$. The corresponding map f is obtained as $f(u,v) = (u+v, 0, 0)$, that is, the image of f is the x-axis in $\mathbb{R}^3$. All points on the uv-plane are singular points.

References

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Exercises

6-1H Consider the equation

$$(*) \qquad (\varphi - \theta)_u = 2a \sin\frac{\varphi+\theta}{2}, \qquad (\varphi+\theta)_v = \frac{2}{a}\sin\frac{\varphi-\theta}{2}$$

for an unknown φ, where $\theta = \theta(u,v)$ is a given function.

(1) Prove that, if θ satisfies the sine-Gordon equation (6.3), φ satisfies the sine Gordon equation, too.

(2) Find the general solution φ of (*) for $\theta = 0$.