7 Surfaces of constant negative curvature—Bäcklund Transformations

Bäcklund transformations. Let $f\colon D \to \mathbb{R}^3$ be a surface with unit normal vector field $\nu\colon D \to \mathbb{R}^3$. A surface $\hat{f}\colon D \to \mathbb{R}^3$ is said to be a *Bäcklund transformation* of f if there exists a positive constant r and an angle δ such that

(B-1) $\hat{f}(p) - f(p)$ is a tangent vector of both the surface f and $\hat{f}$ at $p \in D$,

(B-2) $|f(p) - \hat{f}(p)| = r$,

(B-3) the angle between $\nu(p)$ and $\hat{\nu}(p)$ is δ,

for each $p \in D$.

The following proposition gives a necesary condition for existence of Bäcklund transformations:

Proposition 7.1. *Assume that there exists a Bäcklund transformation $\hat{f}$ of an immersion $f\colon D \to \mathbb{R}^3$. Then r and δ in* (B-1) *and* (B-3) *satisfy*

$$K = -\frac{\sin^2\delta}{r^2},$$

where K is the Gaussian curvature, that is, K is negative constant. Moreover, the Gaussian curvature $\hat{K}$ of the Bäcklund transformation $\hat{f}$ is the same constant as f.

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To have constant negative Gaussian curvature is also the sufficient condition for existence of Bäcklund transformations:

Proposition 7.2. *Let $f\colon D \to \mathbb{R}^3$ be a smooth map with unit normal vector field ν, where D is a simply connected domain of the uv-plane, and*

$$ds^2 := df \cdot df = du^2 + 2\cos\theta\, du\, dv + dv^2,$$
$$I\!I := -df \cdot d\nu = 2\sin\theta\, du\, dv,$$

where $\theta = \theta(u,v)$ is a smooth function satisfying the sine-Gordon equation

$$\theta_{uv} = \sin\theta.$$

We fix $p_0 = (u_0, v_0) \in D$ and $\delta \in (0,\pi)$. Then

(B-1) for any $\varphi_0 \in \mathbb{R}$, there exists a unique solution of the differential equation

$$(\varphi-\theta)_u = 2\cot\frac{\delta}{2}\sin\frac{\varphi+\theta}{2}, \quad (\varphi+\theta)_v = 2\tan\frac{\delta}{2}\sin\frac{\varphi-\theta}{2}$$

with initial condition $\varphi(u_0,v_0) = \varphi_0$,

(B-2) for φ in (B-1), let

$$\hat{f} := f + \sin\delta\left(\cos\frac{\varphi}{2}\boldsymbol{e}_1 + \sin\frac{\varphi}{2}\boldsymbol{e}_2\right)$$
$$\hat{\nu} := \cos\delta\nu - \sin\delta - \sin\delta\left(\left(-\sin\frac{\varphi}{2}\boldsymbol{e}_1 + \cos\frac{\varphi}{2}\boldsymbol{e}_2\right)\right),$$

where

$$\boldsymbol{e}_1 := \frac{1}{2}\sec\frac{\theta}{2}(f_u + f_v), \quad \boldsymbol{e}_2 := \frac{1}{2}\csc\frac{\theta}{2}(-f_u + f_v).$$

Then $\hat{\nu}$ is a unit normal vector field of $\hat{f}\colon D \to \mathbb{R}^3$, and the first and second fundamtal forms are

$$d\hat{f}\cdot d\hat{f} := du^2 + 2\cos\varphi\, du\, dv + dv^2,$$
$$\widehat{II} := 2\sin\varphi\, du\, dv.$$

That is, $\hat{f}$ is a Bäcklund transformation of f, and the asymptotic Chebyshev net of $\hat{f}$ coincides with that of f.

Exercises

7-1[H] Consider the following parametrization of the pseudosphere:

$$\tilde{f}(u,v) = \left(\frac{\cos(u-v)}{\cosh(u+v)}, \frac{\sin(u-v)}{\cosh(u+v)}, (u+v) - \tanh(u+v)\right).$$

- Verify that (u,v) is the asymptotic Chebyshev net.
- Show that $\tilde{f}$ is the Bäcklund transformation of the straigntline

  $$f(u,v) = (0,0,u+v)$$

  accompanied with unit normal

  $$\nu(u,v) = (-\sin(u-v), \cos(u-v), 0).$$