1 Area minimizing surfaces

1.1 A review of surface theory.

Let $D \subset \mathbb{R}^2$ be a domain in the uv-plane and $f\colon D \to \mathbb{R}^3$ an immersion. We often refer to such an immersion as a *surface*. Then the *unit normal vector* of f is given by (with $\pm$-ambiguity)

$$\nu := \frac{f_u \times f_v}{|f_u \times f_v|}\colon D \longrightarrow S^2 = \{\boldsymbol{x} \in \mathbb{R}^3 \,|\, |\boldsymbol{x}| = 1\} \subset \mathbb{R}^3, \tag{1.1}$$

where "$\times$" denotes the vector product of $\mathbb{R}^3$. The *first* and the *second fundamental forms* are defined as

$$\begin{aligned} ds^2 &= df \cdot df = E\,du^2 + 2F\,du\,dv + G\,dv^2, \\ I\!I &= -df \cdot d\nu = L\,du^2 + 2M\,du\,dv + N\,dv^2, \end{aligned} \tag{1.2}$$

where "$\cdot$" denotes the canonical inner product of $\mathbb{R}^3$. Here,

$$\begin{aligned} E &:= f_u \cdot f_u, & F &:= f_u \cdot f_v = f_v \cdot f_u, & G &:= f_v \cdot f_v, \\ L &:= -f_u \cdot \nu_u, & M &:= -f_u \cdot \nu_v = -f_v \cdot \nu_u, & N &:= -f_v \cdot \nu_v \\ &= f_{uu} \cdot \nu, & &= f_{uv} \cdot \nu, & &= f_{vv} \cdot \nu \end{aligned}$$

are called the *entries of the first and the second fundamental forms* with respect to the parameters (u, v). The *area* of the image of a compact region $\Omega \subset D$ is computed as

$$\mathcal{A}(\Omega) := \iint_\Omega dA = \iint_\Omega |f_u \times f_v|\,du\,dv, \tag{1.3}$$

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where $dA = |f_u \times f_v|\,du\,dv = \sqrt{EG - F^2}\,du\,dv$ is said to be the *area element* of the surface.

The derivatives of ν is written as (the Weingarten Formula)

$$\nu_u = -A_1^1 f_u - A_1^2 f_v, \qquad \nu_v = -A_2^1 f_u - A_2^2 f_v, \tag{1.4}$$
$$A := \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

The matrix A is called the *Weingarten matrix*, and the determinant K and the half H of the trace of A are called the *Gaussian curvature* and the *mean curvature*, respectively:

$$K := \det A = \frac{LN - M^2}{EG - F^2}, \qquad H := \frac{1}{2}\operatorname{tr} A = \frac{A_1^1 + A_2^2}{2}. \tag{1.5}$$

1.2 Area minimizing surfaces.

The purpose of this section is to show the following fact:

> For a given simple closed curve C in $\mathbb{R}^3$, the surface which minimizing area among all surfaces bounded by C is a surface whose mean curvature vanishes identically.

Setting up. As the description of the above fact is rather intuituive, we will formulate the problem.

Let C be a simple closed smooth curve in $\mathbb{R}^3$ and set

$$\mathcal{S}_C := \left\{ f\colon \overline{D} \to \mathbb{R}^3\,;\ \begin{matrix} f \text{ is a } C^\infty\text{-immersion} \\ f(\partial D) = C \end{matrix} \right\}, \tag{1.6}$$

where D (resp. $\overline{D}$) is the open (resp. closed) unit disc and ∂D is its boundary:[1]

$$(1.7)\quad \overline{D} := D \cup \partial D, \qquad \begin{aligned} D &: = \{(u,v) \in \mathbb{R}^2\,;\, u^2+v^2<1\},\\ \partial D &: = \{(u,v) \in \mathbb{R}^2\,;\, u^2+v^2=1\}\\ &= \{(\cos\theta, \sin\theta)\,;\, \theta \in \mathbb{R}\}. \end{aligned}$$

Roughly speaking, $\mathcal{S}_C$ is "the set of the surfaces bounded by C". Then we set the *area functional* as

$$(1.8)\qquad \mathcal{A}\colon \mathcal{S}_C \ni f \longmapsto \mathcal{A}(f) = \iint_{\overline{D}} |f_u \times f_v|\, du\, dv.$$

Using these notations, our result can be stated as the following:

Theorem 1.1. *If a surface $f \in \mathcal{S}_C$ attains the minimum of the area functional $\mathcal{A}$, the mean curvature of f vanishes identically.*

Taking this fact into account, we define

Definition 1.2. A surface whose mean curvature vanishes identically is said to be *minimal*.

Remark 1.3. As Theorem 1.1 is a necessary condition for the minimizer, a minimal surface is not necessarily a minimizer of the area functional.

[1] A map f defined on $\overline{D}$ is said to be C^∞ if there exists a open set $\widetilde{D}$ containing $\overline{D}$ and a C^∞ map $\tilde{f}$ defined on $\widetilde{D}$ such that $\tilde{f}|_{\overline{D}} = f$.

Variations of surfaces. To show Theorem 1.1, we want to "differentiate" the functional $\mathcal{A}$.

Definition 1.4. For a surface $f \in \mathcal{S}_C$, a *variation* (fixing the boundary) of f is a C^∞-map

$$\mathcal{F}\colon \overline{D} \times (-\varepsilon, \varepsilon) \ni (u,v;t) \longmapsto f^t(u,v) := \mathcal{F}(u,v;t) \in \mathbb{R}^3$$

such that $f^0 = f$ and $f^t \in \mathcal{S}_C$ for each $t \in (-\varepsilon, \varepsilon)$, where ε is a positive number. The vector-valued function

$$(1.9)\qquad V(u,v) := \left.\frac{\partial}{\partial t}\right|_{t=0} f^t(u,v)$$

is called the *variational vector field* of the variation $\mathcal{F}$.

Lemma 1.5. *For a variation $\mathcal{F} = \{f^t\}$ of $f \in \mathcal{S}_c$ with variational vector field V, it holds that*

$$\frac{d}{d\theta} f(\cos\theta, \sin\theta) \times V(\cos\theta, \sin\theta) = \mathbf{0}.$$

Proof. Since $(\cos\theta, \sin\theta)$ is a parametrization of ∂D, $\gamma^t(\theta) := f^t(\cos\theta, \sin\theta) \in C$ for all t and θ. Thus, two vectors in the left-hand side of the first assertion are both tangent to C, proving the lemma. □

The first variation formula.

Theorem 1.6. *Let $\mathcal{F} = \{f^t\}$ be a variation of $f \in \mathcal{S}_C$ with variational vector field V. Then it holds that*

$$(1.10)\qquad \left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(f^t) = -2 \iint_{\overline{D}} H\big(V \cdot \nu\big)\, dA,$$

where H, ν and dA are the mean curvature, the unit normal vector and the area element of f, respectively.

Proof. By the definition of the area (1.3), we have

$$\begin{aligned}
(*) := \left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(f^t) &= \left.\frac{d}{dt}\right|_{t=0} \iint_{\overline{D}} |f^t_u \times f^t_v| \, du \, dv \\
&= \iint_{\overline{D}} \left.\frac{\partial}{\partial t}\right|_{t=0} |f^t_u \times f^t_v| \, du \, dv \\
&= \iint_{\overline{D}} \frac{\bigl(V_u \times f_v + f_u \times V_v\bigr) \cdot (f_u \times f_v)}{|f_u \times f_v|} \, du \, dv \\
&= \iint_{\overline{D}} \bigl(V_u \times f_v + f_u \times V_v\bigr) \cdot \nu \, du \, dv \\
&= \iint_{\overline{D}} \bigl(\bigl(V_u \times f_v\bigr) \cdot \nu + \bigl(f_u \times V_v\bigr) \cdot \nu\bigr) \, du \, dv.
\end{aligned}$$

Here, by the formula of *scalar triple product*

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = (\boldsymbol{b} \times \boldsymbol{c}) \cdot \boldsymbol{a} = (\boldsymbol{c} \times \boldsymbol{a}) \cdot \boldsymbol{b} = \det(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}),$$

we have

$$\begin{aligned}
(*) &= \iint_{\overline{D}} \bigl(\bigl(\nu \times f_v\bigr) \cdot V_u + \bigl(f_u \times \nu\bigr) \cdot V_v\bigr) \, du \, dv \\
&= (\mathrm{I}) - (\mathrm{II}), \\
(\mathrm{I}) &:= \iint_{\overline{D}} \Bigl[\bigl(\bigl(\nu \times f_v\bigr) \cdot V\bigr)_u + \bigl(\bigl(f_u \times \nu\bigr) \cdot V\bigr)_v\Bigr] \, du \, dv, \\
(\mathrm{II}) &:= \iint_{\overline{D}} \Bigl[\bigl(\bigl(\nu \times f_v\bigr)_u \cdot V\bigr) + \bigl(\bigl(f_u \times \nu\bigr)_v \cdot V\bigr)\bigr)\Bigr] \, du \, dv.
\end{aligned}$$

By the Green-Stokes formula, (I) is computed as

$$\begin{aligned}
(\mathrm{I}) &= \iint_{\overline{D}} \Bigl[\bigl(\bigl(\nu \times f_v\bigr) \cdot V\bigr)_u - \bigl(\bigl(\nu \times f_u\bigr) \cdot V\bigr)_v\Bigr] \, du \, dv, \\
&= \int_{\partial D} \nu \cdot \bigl(\bigl(f_u \, du + f_v \, dv\bigr) \times V\bigr) \\
&= \int_{-\pi}^{\pi} \nu \cdot \left(\frac{d}{d\theta} f(\cos\theta, \sin\theta) \times V(\cos\theta, \sin\theta)\right) d\theta = 0.
\end{aligned}$$

Here, the last assertion is obtained by Lemma 1.5. On the other hand, using the Weingarten formula (1.4), (II) is computed as

$$\begin{aligned}
(\mathrm{II}) :&= \iint_{\overline{D}} \Bigl[\bigl(\nu_u \times f_v\bigr) \cdot V + \bigl(\nu \times f_{vu}\bigr) \cdot V \\
&\qquad + \bigl(f_{uv} \times \nu\bigr) \cdot V + \bigl(f_u \times \nu_v\bigr) \cdot V\Bigr] \, du \, dv \\
&= \iint_{\overline{D}} \Bigl[\bigl(\nu_u \times f_v\bigr) \cdot V + \bigl(f_u \times \nu_v\bigr) \cdot V\Bigr] \, du \, dv \\
&= -\iint_{\overline{D}} \Bigl[\bigl((A^1_1 f_u + A^2_1 f_v) \times f_v\bigr) \cdot V \\
&\qquad + \bigl(f_u \times (A^1_2 f_u + A^2_2 f_v)\bigr) \cdot V\Bigr] \, du \, dv \\
&= -\iint_{\overline{D}} (A^1_1 + A^2_2)(f_u \times f_v) \cdot V \, du \, dv \\
&= -\iint_{\overline{D}} 2H(\nu \cdot V)|f_u \times f_v| \, du \, dv \qquad \square
\end{aligned}$$

Proof of Theorem 1.1. We need the following "the fundamental lemma for calculus of variations".

Lemma 1.7. *Assume a smooth function $h\colon \overline{D} \to \mathbb{R}$ satisifes*

$$\iint_{\overline{D}} h(u,v)\varphi(u,v)\,du\,dv = 0$$

for all C^∞-function with $\varphi|_{\partial D} = 0$. Then $h = 0$ on D.

Proof. Assume $h(u_0,v_0) > 0$ (resp. < 0)($(u_0,v_0) \in D$). By a continuity, there exists $\varepsilon > 0$ such that $h(u,v) > -$ on an ε-ball $B := B_\varepsilon(u_0,v_0)$ centered at (u_0,v_0). Let φ be a non-negative C^∞-function on $\overline{D}$ such that $\varphi > 0$ on B and 0 on $\overline{D} \setminus B$. Then

$$\iint_{\overline{D}} h\varphi\,du\,dv = \iint_B h\,\varphi\,du\,dv > 0 \qquad \text{(resp. } < 0\text{)},$$

a contradiction. □

Proof of Theorem 1.6. Assume $f \in \mathcal{S}_C$ minimizes the area. Then for any variation $\mathcal{F} = \{f^t\}$ of f, $\mathcal{A}(f^t)$ is not less than $\mathcal{A}(f) = \mathcal{A}(f^0)$. Then by Theorem 1.6, it holds that

$$0 = \left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(f^t) = -2\int_{\overline{D}} H(V\cdot\nu)|f_u \times f_v|\,du\,dv.$$

Let φ be a C^∞-function on $\overline{D}$ with $\varphi|_{\partial D} = 0$. Then $f^t := f + t\varphi\nu$ is a variation of f with variational vector field $V = \varphi\nu$. Thus,

$$\iint H|f_u \times f_v|\varphi = 0.$$

Since φ is arbitrary, Lemma 1.7 yields the conclusion. □

Exercises

1-1$^{\mathrm{H}}$ For P, Q $\in \mathbb{R}^2$, set

$$\mathcal{C}_{\mathrm{P,Q}} := \left\{\gamma\colon [0,1] \to \mathbb{R}^2;\ \begin{matrix}\gamma \text{ is a regular curve} \\ \gamma(0) = \mathrm{P},\ \gamma(1) = \mathrm{Q}\end{matrix}\right\},$$

and denote by $\mathcal{L}$ the length functional:

$$\mathcal{L}(\gamma) := \int_0^1 |\dot\gamma(s)|\,ds \qquad \left(\dot{} = \frac{d}{ds}\right)$$

A variation of a curve $\gamma \in \mathcal{C}_{\mathrm{P,Q}}$ is a C^∞-map

$$\mathit{\Gamma}\colon [0,1] \times (-\varepsilon,\varepsilon) \to \gamma^t(s) = \mathit{\Gamma}(s,t) \in \mathbb{R}^2$$

such that $\gamma^t \in \mathcal{C}_{\mathrm{P,Q}}$ for each $t \in (-\varepsilon,\varepsilon)$ and $\gamma^0 = \gamma$.

Then show the first variation formula for the length functional

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{L}(\gamma^t) = -\int_0^1 (V\cdot\boldsymbol{h})\,ds, \qquad \boldsymbol{h} := \frac{\ddot y\dot x - \ddot x\dot y}{|\dot\gamma|^3}(-\dot y, \dot x),$$

where V is the variational vector field of the variation $\{\gamma^t\}$ of the curve $\gamma(s) = (x(s), y(s))$.