2 Classical Examples

Graphs. For a C^∞ function $\varphi(x,y)$ on a domain (or an open set) $D \subset \mathbb{R}^2$, its graph is considered as a parametrized surface

$$(2.1) \qquad f\colon D \ni (x,y) \longmapsto \bigl(x, y, \varphi(x,y)\bigr) \in \mathbb{R}^3.$$

The surface (2.1) is minimal if and only if

$$(2.2) \quad (2\delta^3 H =) \quad (1+\varphi_y^2)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} + (1+\varphi_x^2)\varphi_{yy} = 0,$$

where $\delta = \sqrt{1+\varphi_x^2+\varphi_y^2}$. The (nonlinear, elliptic) partial differential equation (2.2) is called the *minimal surface equation*.

Example 2.1. A linear function $\varphi(x,y) = ax+by+c$ (a, b and c are constants) satisfies (2.2), and its graph is a plane. It is known that the *entire* (i.e., defined on whole $\mathbb{R}^2$) solution of (2.2) is a linear function (Bernstein [2-1], [2-2]).

Example 2.2. The graph of the function

$$(2.3) \quad \varphi(x,y) = \frac{1}{a}\log\frac{\cos ay}{\cos ax} \qquad (a>0 \text{ is a constant})$$

$$(x,y) \in \bigcup_{\substack{m,\,n\in\mathbb{Z}\\ m+n\colon \text{even}}} \left\{(x,y)\in\mathbb{R}^2 \;\middle|\; |ax-m\pi| < \tfrac{\pi}{2}\,,\ |ay-n\pi| < \tfrac{\pi}{2}\right\}$$

is a minimal surface, called the *Scherk surface* (Figure 1). On the domain $\{(x,y); |ax|<\pi/2, |ay|<\pi/2\}$, φ is expressed as

$$\varphi(x,y) = \tfrac{1}{a}\log\cos ax - \tfrac{1}{a}\log\cos ay.$$

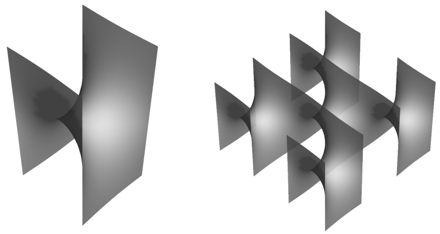

Figure 1: the Scherk surface

In general, a graph of a function $\varphi(x,y) = F(x)+G(y)$ is called a *translation surface*.

Theorem 2.3. *A translation minimal surface is congruent to a part of a plane or a part of the Scherk surface.*

Proof. For $\varphi(x,y) = F(x)+G(y)$, (2.2) is equivalent to

$$(2.4) \qquad \frac{F''}{1+(F')^2} = -\frac{\ddot G}{1+(\dot G)^2} =: a.$$

Since the left-hand (resp. middle) side of (2.4) is a function depending only on x (resp. y), a must be a constant. When $a=0$, (2.4) reduce to $F''=0$, $\ddot G = 0$, i.e., φ is a linear function.

Assume $a \neq 0$. Without loss of generality, we may assume that $a>0$ Then the first equation in (2.4) yields $\tan^{-1} F'(x) = ax + c_1$, where c_1 is a constant. By a translation along the x-axis, we can set $c_1 = 0$, and then $F(x) = -\frac{1}{a}\log\cos ax + c_2$,

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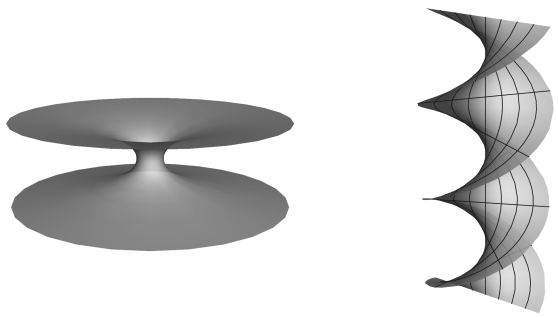

Figure 2: The catenoid and the helicoid.

with constant c_2. By a translation along the z-axis, we may set $c_2 = 0$: $F(x) = -\frac{1}{a}\log\cos ax$. Similarly, we have $G(y) = \frac{1}{a}\log\cos ay$. □

Surfaces of revolution. We consider a *surface of revolution*

$$\text{(2.5)}\quad \begin{aligned} f(u,v) &= \bigl(x(u)\cos v, x(u)\sin v, z(u)\bigr),\\ \gamma(u) &:= \bigl(x(u), z(u)\bigr)\colon \mathbb{R}\supset I \to \mathbb{R}^2, \qquad x(u)\neq 0 \end{aligned}$$

where γ is a regular curve on the xz-plane, called the *profile curve* of the surface of revolution.

Example 2.4. Let $\gamma(u) = (a\cosh\frac{u}{a}, u)$, that is, γ is the graph $x = a\cosh\frac{z}{a}$ on the xz-plane, called the *catenary*. Then the surface (2.5) is minimal, called *catenoid* (Figure. 2, left).

Theorem 2.5. *A minimal surface of revolution is congruent to a part of the catenoid or the plane.*

Proof. We assume that $x(u) > 0$ and u in (2.5) is the arclength parameter of γ:

$$\text{(2.6)}\qquad (x')^2 + (z')^2 = 1 \qquad ({}' = d/du).$$

Then f is minimal if and only if

$$\text{(2.7)}\qquad 2H = x'z'' - z'x'' + \frac{z'}{x} = 0.$$

We shall determine $\bigl(x(u), z(u)\bigr)$ satisfying (2.7) and (2.6).

Assume $\bigl(x(u), z(u)\bigr)$ satisfy these equations and consider the case that $z'\neq 0$ for some interval I'. By a reflection about the x-axis, we may assume $z' > 0$ on I'. Differentiating (2.6), we have $x'x'' + z'z'' = 0$. Hence, noticing z' is positive on I', (2.7) is equivalent to

$$\begin{aligned} 0 &= z'\left(x'z'' - z'x'' + \frac{z'}{x}\right) = x'z'z'' - (z')^2x' + \frac{(z')^2}{x}\\ &= -x'x'x'' - \bigl(1-(x')^2\bigr)x'' + \frac{1-(x')^2}{x} = x'' + \frac{1-(x')^2}{x}. \end{aligned}$$

Since $1-(x')^2 = (z')^2 > 0$ and $x > 0$, this is equivalent to

$$\frac{-2x'x''}{1-(x')^2} = \frac{-2x'}{x}.$$

Integrating this in u, we have

$$\log\bigl(1-(x')^2\bigr) = \log x^{-2} + \text{constant}, \text{ that is,} \quad 1-(x')^2 = \frac{a^2}{x^2},$$

where a is a constant. Hence we have

$$x' = \pm\sqrt{1-\frac{a^2}{x^2}}, \quad \text{that is,} \quad du = \frac{\pm x\,dx}{\sqrt{x^2-a^2}}.$$

Integrating this, we get $\sqrt{x^2-a^2} = \pm u+$constant. By a change of the arclength parameter $u \mapsto \pm u + \text{constant}$, we have

$$u = \sqrt{x^2-a^2}, \quad \text{i.e.,} \quad x = \sqrt{u^2+a^2}. \tag{2.8}$$

By (2.6) and the assumption $z' > 0$, we have $z' = a/\sqrt{u^2+a^2}$, and

$$z = \int \frac{a}{\sqrt{u^2+a^2}}\,du = a\log\left(u+\sqrt{u^2+a^2}\right) + \text{constant}.$$

By a translation along the z-axis, we may choose the constant above to be $-a\log a$. Then we have

$$z = a\log\bigl((u+\sqrt{u^2+a^2})/a\bigr), \tag{2.9}$$

and thus, $\cosh\frac{z}{a} = \frac{1}{a}\sqrt{u^2+a^2} = \frac{x}{a}$. Therefore, the curve $\bigl(x(u), z(u)\bigr)$ is a catenary, and z' does not vanish on whole I.

Otherwise, if $z' = 0$ on an interval I, $z(u)$ is constant. Thus the corresponding surface is a part of horizontal plane. □

Ruled surfaces. Let $\gamma(u)$ be a parametrized space curve, and $\xi(u)$ is a vector valued function such that $\dot\gamma(u)$, and $\xi(u)$ are linearly independent for each u. Then a parametrized surface

$$f(u,v) := \gamma(u) + v\xi(u) \tag{2.10}$$

is called a *ruled surface*, because it is a locus of moving straight lines. Replacing ξ by $\xi/|\xi|$ and $v|\xi|$ by v, we may assume without loss of generality that $|\xi| = 1$. Moreover, if we set

$$\tilde\gamma(u) := \gamma(u) + \tau(u)\xi(u), \qquad \tau(u) := \int_{u_0}^{u} \dot\gamma(t)\cdot\xi(t)\,dt, \tag{2.11}$$

(2.10) is written as $\tilde\gamma(u) + \tilde v\xi(u)$ ($\tilde v = v - \tau$), where $\tilde\gamma'\cdot\xi = 0$. Finally, we can choose u to be the arclength of γ.

Summing up, any ruled surface can be expressed as

$$f(u,v) = \gamma(u) + v\xi(u), \qquad |\xi(u)| = |\gamma'(u)| = 1, \quad \gamma'(u)\cdot\xi(u) = 0. \tag{2.12}$$

Example 2.6. For $\gamma(u) := (0,0,u)$ and $\xi(u) := (\cos au, \sin au, 0)$ ($a > 0$ is a constant), the surface (2.10) is minimal, called the *helicoid* (Figure 2, right).

Theorem 2.7. *A minimal ruled surface is congruent to a part of a helicoid or a plane.*

Proof. Assume that (2.12) is minimal. Since $\xi\cdot\xi' = 0$, entries of the first and the second fundamental forms satisfy $F := f_u\cdot f_v = 0$ and $N := f_{vv}\cdot\nu = 0$. Thus, f is minimal if and only if

$$2\sqrt{EG-F^2}^3 H = EN - 2FM + GL = GL = 0, \text{ i.e.} \quad L = 0.$$

Since

$$|f_u\times f_v|L = (f_u\times f_v)\cdot f_{uu} \quad = \det(\gamma' + v\xi', \xi, \gamma'' + v\xi''),$$

the condition $H = 0$ is equivalent to

$$\det(\gamma', \xi, \gamma'') = 0, \tag{2.13}$$

$$\det(\xi', \xi, \gamma'') + \det(\gamma', \xi, \xi'') = 0, \tag{2.14}$$

$$\det(\xi', \xi, \xi'') = 0. \tag{2.15}$$

Here, $\{\gamma', \xi, \gamma' \times \xi\}$ forms an orthonormal basis of $\mathbb{R}^3$ for each u satisfying the following Frenet-Serret-type formulas:

$$\gamma'' = \kappa\xi, \quad \xi' = -\kappa\gamma' + \tau(\gamma' \times \xi), \quad (\gamma' \times \xi)' = -\tau\xi, \tag{2.16}$$

where κ and τ are smooth functions in u. In fact, since $|\gamma'| = 1$, $\gamma'' \cdot \gamma' = 0$, and (2.13) implies $\gamma'' \cdot (\gamma' \times \xi) = 0$. Thus the first equation follows. Similarly, $\xi' \cdot \xi = 0$ and $\xi' \cdot \gamma' = (\xi \cdot \gamma')' - \xi \cdot \gamma'' = -\xi \cdot \gamma'' = -\kappa$ yield the second equation. Finally,

$$(\gamma' \times \xi)' \cdot \gamma' = -(\gamma' \times \xi) \cdot \gamma'' = 0, \quad (\gamma' \times \xi)' \cdot \xi = -(\gamma' \times \xi) \cdot \xi' = -\tau$$

imply the third equation.

Differentiating (2.14) with (2.16), we have

$$\xi'' = -\kappa'\gamma' - (\kappa^2 + \tau^2)\xi + \tau'(\gamma' \times \xi). \tag{2.17}$$

Hence (2.14), $0 = \det(\gamma', \xi, \xi'') = \tau'$, and then τ is constant. In addition, by (2.15), we have

$$0 = \det(\xi', \xi, \xi'') = (-\kappa\tau' + \kappa'\tau) = \det(\gamma', \xi, \gamma' \times \xi) = \kappa'\tau.$$

Assume the constant $\tau \neq 0$. Then $\kappa' = 0$, that is, κ is also constant, and (2.17) turns to be

$$\xi'' = -(\kappa^2 + \tau^2)\xi. \tag{2.18}$$

So, if we set $\tilde{\gamma} := \gamma + (\kappa^2 + \tau^2)\xi$ and $\tilde{v} = v - (\kappa^2 + \tau^2)$, we have $f = \tilde{\gamma} + \tilde{v}\xi$ with $\tilde{\gamma}'' = 0$, that is, $\tilde{\gamma}$ is a straight line. Then by an isometry of $\mathbb{R}^3$ and a change of parameter u, we can set $\tilde{\gamma}(u) = (0, 0, u)$. Since ξ is perpendicular to $\tilde{\gamma}' = (0, 0, 1)$, the image of $\xi(u)$ lies on the unit circle in the xy-plane. Hence, by (2.18), up to an isometry and a change of parameters, we have

$$\xi(u) = (\cos au, \sin au, 0), \qquad a = \sqrt{\kappa^2 + \tau^2} > 0,$$

Then the surface is a helicoid.

On the other hand, when $\tau = 0$, $\gamma' \times \xi$ is constant, and we may set $\gamma' \times \xi = (0, 0, 1)$. Since γ' and ξ are perpendicular to $(0, 0, 1)$, $f(u, v) = \gamma(u) + v\xi(u)$ lies on a plane parallel to the xy-plane, that is, the image of the surface is part of a plane. □

References

[2-1] Bernstein, S. N., *Sur une théorme de géometrie et ses applications aux équations dérivées partielles du type elliptique*, Comm. Soc. Math. Kharkov **15** 38–45. (1915–1917).

[2-2] Osserman, R., A survey of minimal surfaces, Dover Publ.

Exercises

2-1$^{\mathrm{H}}$ Show that the surface $\{(x, y, z)\,;\, \sinh x \sinh y = \sin z\}$ is minimal.