3 Isothermal Coordinates

A Review of Complex Analysis. Let $\mathbb{C}$ be the complex plane. A C^1-function[2] $f\colon \mathbb{C} \ni D \in z \mapsto w = f(z) \in \mathbb{C}$ defined on a domain D is said to be *holomorphic* if the derivative

$$f'(z) := \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists for all $z \in D$.

Fact 3.1 (The Cauchy-Riemann equation)**.** *A function $f\colon \mathbb{C} \ni D \to \mathbb{C}$ is holomorphic if and only if*

$$\frac{\partial u}{\partial \xi} = \frac{\partial v}{\partial \eta} \qquad \textit{and} \qquad \frac{\partial u}{\partial \eta} = -\frac{\partial v}{\partial \xi} \tag{3.1}$$

holds on D, where $w = f(z)$, $z = \xi + i\eta$, $w = u + iv$ ($i = \sqrt{-1}$).

For functions of complex variable $z = \xi + i\eta$, we set

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial \xi} - i\frac{\partial}{\partial \eta}\right), \quad \frac{\partial}{\partial \bar z} := \frac{1}{2}\left(\frac{\partial}{\partial \xi} + i\frac{\partial}{\partial \eta}\right). \tag{3.2}$$

Corollary 3.2. *For a complex function f,* (3.1) *is equivalent to*

$$\frac{\partial f}{\partial \bar z} = 0. \tag{3.3}$$

Proof. Setting $w = f(z) = u + iv$ and $z = \xi + i\eta$. Then the real (resp. imaginary) part of the left-hand side of (3.3) coincides with the first (resp. second) equation of (3.1). □

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[2] Of class C^1 as a map from $D \subset \mathbb{R}^2$ to $\mathbb{R}^2$.

Definition 3.3. A real-valued function $\varphi\colon \mathbb{R}^2 \supset U \to \mathbb{R}$ is said to be *harmonic* if it satisfies the Laplace equation

$$\Delta\varphi := \varphi_{\xi\xi} + \varphi_{\eta\eta} = 0.$$

Lemma 3.4. *If a function $\varphi\colon \mathbb{C} \supset D \to \mathbb{R}$ is harmonic, $\partial\varphi/\partial z$ is a holomorphic function on D, where z is a complex coordinate of $\mathbb{C}$.*

Proof. Corollary 3.2 yields the conclusion since

$$\frac{\partial}{\partial \bar z}\frac{\partial \varphi}{\partial z} = \frac{\partial^2 \varphi}{\partial \bar z \partial z} = \frac{1}{4}\Delta\varphi. \qquad \square$$

Isothermal Coordinates.

Definition 3.5. Let $f\colon M^2 \to \mathbb{R}^3$ be an immersion of 2-manifold, and ds^2 its first fundamental form. A local coordinate chart $\big(U; (u,v)\big)$ of M^2 is called an *isothermal coordinate system* or a *conformal coordinate system* if ds^2 is written in the form[3]

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \qquad \sigma = \sigma(u,v) \in C^\infty(U).$$

Example 3.6. A parametrization of the catenoid in Example 2.4 is isothermal if $a = 1$. In fact, the first fundamental form is expressed as $\cosh^2(u/a)(du^2 + a^2 dv^2)$.

[3] The notion of the isothermal coordinate system can be defined not only for surfaces but also for Riemannian 2-manifolds, that is, differentiable 2-manifolds M^2 with Riemannian metrics ds^2 (the first fundamental forms).

Definition 3.7. Two charts $(U_j; (u_j, v_j))$ $(j = 1, 2)$ of a 2-manifold M^2 has the *same* (resp. *opposite*) *orientation* if the Jacobian $\frac{\partial(u_2,v_2)}{\partial(u_1,v_1)}$ is positive (resp. negative) on $U_1 \cap U_2$. A manifold M^2 is said to be *oriented* if there exists an atlas $\{(U_j; (u_j, v_j))\}$ such that all charts have the same orientations. A choice of such an atlas is called an *orientation* of M^2.

Proposition 3.8. *Let (u, v) be an isothermal coordinate system of a surface. Then another coordinate system (ξ, η) is also isothermal if and only if the parameter change $(\xi, \eta) \mapsto (u, v)$ satisfy*

$$\frac{\partial u}{\partial \xi} = \varepsilon \frac{\partial v}{\partial \eta}, \qquad \frac{\partial u}{\partial \eta} = -\varepsilon \frac{\partial v}{\partial \xi}, \tag{3.4}$$

where $\varepsilon = 1$ (resp. -1) if (u, v) and (ξ, η) has the same (resp. the opposite) orientation.

Proof. If we write $ds^2 = e^{2\sigma}(du^2 + dv^2)$, it holds that

$$ds^2 = e^{2\sigma}\big((u_\xi^2 + v_\xi^2)\, d\xi^2 + 2(u_\xi v_\eta + u_\eta v_\xi)\, d\xi\, d\eta + (u_\eta^2 + v_\eta^2)\, d\eta^2\big).$$

Thus, (ξ, η) is isothermal if and only if

$$u_\xi^2 + v_\xi^2 = u_\eta^2 + v_\eta^2, \qquad (u_\xi v_\eta + u_\eta v_\xi) = 0. \tag{3.5}$$

The second equality yields $(v_\xi, v_\eta) = \varepsilon(-u_\eta, u_\xi)$ for some function ε. Substituting this into the first equation of (3.5), we get $\varepsilon = \pm 1$. Moreover,

$$\frac{\partial(u, v)}{\partial(\xi, \eta)} = \det \begin{pmatrix} u_\xi & u_\eta \\ v_\xi & v_\eta \end{pmatrix} = \det \begin{pmatrix} u_\xi & u_\eta \\ -\varepsilon u_\eta & \varepsilon u_\xi \end{pmatrix} = \varepsilon(u_\xi^2 + u_\eta^2).$$

Thus, the conclusion follows. □

Corollary 3.9. *Let (u, v) is an isothermal coordinate system. Then a coordinate system (ξ, η) is isothermal and has the same orientation as (u, v) if and only if the map $\xi + i\eta \mapsto u + iv$ $(i = \sqrt{-1})$ is holomorphic.*

Proof. Equations 3.4 for $\varepsilon = +1$ are nothing but the Cauchy-Riemann equations (3.1). □

Fact 3.10 (Section 15 in [3-1])**.** *Let (M^2, ds^2) be an arbitrary Riemannian manifold. Then for each $p \in M^2$, there exists an isothermal chart containing p.*

Corollary 3.11. *Any oriented Riemannian 2-manifold (M^2, ds^2) has a structure of Riemann surface (i.e., a complex 1-manifold) such that for each complex coordinate $z = u + iv$, (u, v) is an isothermal coordinate system for ds^2.*

Proof. Let $p \in M^2$ and take a local coordinate chart $(U_p; (x, y))$ at p which is compatible to the orientation of M^2. Then by Fact 3.10, their exists a isothermal coordinate system $(V_p; (u_p, v_p))$ at p. Moreover, replacing (u, v) by (v, u) if necessary, we can take (u, v) which has the same orientation of (x, y). Thus, we have an atlas $\{(V_p; (u_p, v_p))\}$ consists of isothermal coordinate systems. Since each chart is compatible of the orientation, the coordinate change $z_p = u_p + iv_p \mapsto u_q + iv_q = z_q$ is holomorphic. Hence we get a complex atlas $\{(V_p; z_p)\}$. □

Isothermal Coordinates for Minimal surfaces. Though existence of isothermal parameters are guaranteed as Fact 3.10, we shall give an alternative proof of it for minimal surfaces. The proof is due to [3-2].

Lemma 3.12 (The Poincaré lemma [Theorem 12.2 in [3-1]]). *Let $D \subset \mathbb{R}^2$ be a simply connected domain, and let λ, μ be smooth functions defined on D. If*

$$\lambda_\xi = \mu_\eta, \qquad \text{that is} \qquad d\omega = 0 \quad \text{for} \quad \omega = \lambda\, d\xi + \mu\, d\eta,$$

then there exists a smooth function α on D such that

$$\alpha_\xi = \lambda, \quad \alpha_\eta = \mu, \qquad \text{that is,} \qquad d\alpha = \omega.$$

Proposition 3.13. *Assume that the graph of $\varphi\colon D_R \to \mathbb{R}$ defined on a disc $D_R := \{(x,y)\,;\, x^2+y^2 < R^2\}$ is minimal surface. Then there exists smooth map*

$$X\colon D_R \ni (x,y) \longmapsto \bigl(\xi(x,y), \eta(x,y)\bigr) \in X(D_R) \subset \mathbb{R}^2$$

such that

1. *$X\colon D_R \to X(D_R)$ is a diffeomorphism with $X(\mathbf{0}) = \mathbf{0}$,*
2. *(ξ, η) is an isothermal parameter of the graph $z = \varphi(x,y)$.*
3. *$X(D_R) \supset \{(\xi,\eta)\,;\, \xi^2+\eta^2 < R^2\}$.*

Proof. By the assumption, φ satisfies (2.2):

$$(1+\varphi_x^2)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} + (1+\varphi_y^2)\varphi_{yy} = 0. \tag{3.6}$$

Let $W := \sqrt{1+\varphi_x^2+\varphi_y^2}$ and set

$$\lambda_1 := \frac{1+\varphi_x^2}{W}, \quad \mu_1 = \lambda_2 := \frac{\varphi_x\varphi_y}{W}, \quad \mu_2 := \frac{1+\varphi_y^2}{W}. \tag{3.7}$$

So one can show that $(\lambda_1)_y = (\mu_1)_x$ and $(\lambda_2)_y = (\mu_2)_x$. Then by Lemma 3.12, there exist smooth functions α, β such that

$$\alpha_x = \lambda_1, \quad \alpha_y = \mu_1, \quad \beta_x = \lambda_2, \quad \beta_y = \mu_2.$$

Adding constants, we may assume $\alpha(0,0) = \beta(0,0) = 0$. Using these, we define a map $X = (\xi,\eta)\colon D_R \to \mathbb{R}^2$ by

$$\xi(x,y) := x + \alpha(x,y), \qquad \eta(x,y) := y + \beta(x,y). \tag{3.8}$$

By definition, the Jacobian of X is computed as

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \det\begin{pmatrix} 1+\lambda_1 & \mu_1 \\ \lambda_2 & 1+\mu_2 \end{pmatrix} = 2(2+\varphi_x^2+\varphi_y^2) > 0.$$

Hence X is a local diffeomorphism. So, to prove (1), it is sufficient to show that X is injective: Fix $\boldsymbol{x}_0 = (x_0,y_0) \in D_R$ and $\boldsymbol{h} = (h,k)$ such that $\boldsymbol{x}_1 := \boldsymbol{x}_0 + \boldsymbol{h} \in D_R$. We set $\boldsymbol{x}_t := \boldsymbol{x} + t\boldsymbol{h}$ $(0 \leqq t \leqq 1)$, $\boldsymbol{X}_t := X(\boldsymbol{x}_t)$, $\boldsymbol{\alpha}_t := \bigl(\alpha(\boldsymbol{x}_t), \beta(\boldsymbol{x}_t)\bigr)$, and

$$q(t) := \boldsymbol{h}\cdot(\boldsymbol{\alpha}_t - \boldsymbol{\alpha}_0) \quad (0 \leqq t \leqq 1).$$

Then by the mean value theorem, it holds that

$$\begin{aligned}\boldsymbol{h}\cdot(\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_0) &= q'(\tau) = \boldsymbol{h}\cdot\boldsymbol{\alpha}'(\tau) = h^2\lambda_1 + hk(\mu_1+\lambda_2) + k^2\mu_2 \\ &= W^{-1}\bigl((1+\varphi_x^2)h^2 + 2\varphi_x\varphi_y hk + (1+\varphi_y^2)k^2\bigr) > 0\end{aligned}$$

for some $\tau \in (0,1)$, because the quadratic form in (h,k) of the right-hand side is positive definite. Hence

$$\begin{aligned}|X(\boldsymbol{x}_0+\boldsymbol{h}) - X(\boldsymbol{x}_0)|^2 &= |\boldsymbol{x}_1 - \boldsymbol{x}_0 + \boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_0|^2 \\ &= |\boldsymbol{h}|^2 + 2\boldsymbol{h}\cdot(\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_0) + |\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_0|^2 \geqq |\boldsymbol{h}|^2,\end{aligned} \tag{3.9}$$

which proves the injectivity of X.

By definition, $d\xi = (1+\lambda_1)\,dx + \mu_1\,dy$, and $d\eta = \lambda_2\,dx + (1+\mu_2)\,dy$ hold. So,

$$(3.10)\quad d\xi^2 + d\eta^2 = \left(1+\frac{1}{W}\right)^2 ds^2,$$
$$ds^2 = (1+\varphi_x^2)\,dx^2 + 2\varphi_x\varphi_y\,dx\,dy + (1+\varphi_y^2)\,dy^2,$$

proving (2).

Finally, we prove (3). Let $\rho := \inf\{|\boldsymbol{X}|\,|\,\boldsymbol{X}\in X(D_R)^c\}$. Then $\rho > 0$ because X is a diffeomorphism and $X(\mathbf{0}) = \mathbf{0}$. Since the result is obvious if $\rho = +\infty$, we consider the case $\rho \in (0,\infty)$. The set $X(D_R)^c$ is a closed subset in $\mathbb{R}^2$ because X is a diffeomorphism. Hence there exists $\boldsymbol{X}_\rho \in X(D_R)^c$ with $|\boldsymbol{X}_\rho| = \rho$. Since $\boldsymbol{X}_\rho \in \partial X(D_R)^c = \partial X(D_R)$, there exists a sequence $\{\boldsymbol{X}_n\}\subset X(D_R)$ which convergences to $\boldsymbol{X}_\rho$. The inverse image of $\{\boldsymbol{x}_n := X^{-1}(\boldsymbol{X}_n)\}$ of such a sequence is a sequence in D_R, which does not accumlate in D_R. Hence, by taking a subsequence if necessary, $\{\boldsymbol{x}_n\}$ converges to $\boldsymbol{x}_R \in \partial D_R$, that is, $|\boldsymbol{x}_R| = R$. Here, setting $\boldsymbol{x}_0 = (0,0)$ in (3.9), we have $|\boldsymbol{x}_n| \leqq |\boldsymbol{X}_n|$, and then, $|\boldsymbol{X}_\rho| \geqq R$, that is, $X(D_R)^c \subset D_R^c$, proving (3). □

The minimal surface equation. The equation for minimal surfaces are linearlized by the isothermal coordinate system:

Proposition 3.14. *Let $f\colon \mathbb{R}^2 \supset D \to \mathbb{R}^3$ be a surface, and assume the parameter (u,v) is isothermal. Then f is minimal if and only if $\Delta f = f_{uu} + f_{vv} = 0$.*

Proof. Write the first fundamental form as $ds^2 = e^{2\sigma}(du^2+dv^2)$. Then $f_u\cdot f_u = f_v\cdot f_v = e^{2\sigma}$ and $f_u\cdot f_v = 0$ hold. So

$$f_{uu}\cdot f_u = \frac{1}{2}(f_u\cdot f_u)_u = \sigma_u e^{2\sigma},$$
$$f_{vv}\cdot f_u = (f_v\cdot f_u)_v - f_v\cdot f_{vu} = -\frac{1}{2}(f_v\cdot f_v)_u = -\sigma_u e^{2\sigma},$$

that is $(f_{uu}+f_{vv})\cdot f_u = 0$. Similarly, one can show $(f_{uu}+f_{vv})\cdot f_v = 0$ and hence $f_{uu}+f_{vv}$ is parallel to the unit normal vector ν. On the other hand, the mean curvature H is computed as

$$H = \frac{L+N}{2e^{2\sigma}} = \frac{(f_{uu}+f_{vv})\cdot\nu}{2e^{2\sigma}}, \quad \text{that is,} \quad \Delta f = 2He^{2\sigma}\nu. \qquad \square$$

References

[3-1] 梅原雅顕・山田光太郎：曲線と曲面—微分幾何的アプローチ（改訂版）.

[3-2] Osserman, R., A SURVEY OF MINIMAL SURFACES, Dover Publ.

Exercises

3-1[H] Consider two minimal surfaces

$$f(u,v) = (\cosh u\cos v, \cosh u\sin v, u),$$
$$g(s,t) = (s\cos t, s\sin t, t).$$

(1) Show that (u,v) is an isothermal parameter of f.

(2) Show that there exists a isothermal parameter (u,v) of g.