4 Bernstein's Theorem

More complex analysis.

Theorem 4.1 (Liouville's theroem)**.** *A bounded holomorphic function defined on the whole complex plane $\mathbb{C}$ is constant.*

Proof. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that $|f(z)| \leqq M$ for every $z \in \mathbb{C}$. Fix a point $z \in \mathbb{C}$. Then by Cauchy's integral formula, it holds that

$$f'(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(\zeta)\,d\zeta}{(z-\zeta)^2} \quad (C_R : \zeta = z + Re^{i\theta}; -\pi < \theta \leqq \pi),$$

where R is an arbitrary positive number. Hence

$$\begin{aligned} |f'(z)| &\leqq \frac{1}{2\pi}\int_{C_R} \frac{|f(\zeta)|\,|d\zeta|}{|z-\zeta|^2} \\ &\leqq \frac{1}{2\pi}\int_{C_R} \frac{M\,|d\zeta|}{|z-\zeta|^2} = \frac{1}{2\pi}\int_{\pi}^{\pi} \frac{M\,R\,d\theta}{R^2} = \frac{M}{R}. \end{aligned}$$

Since R is arbitrary, we can conclude $f'(z) = 0$ by letting $R \to \infty$. Moreover, since z is arbitrary, $f'(z) = 0$ holds on $\mathbb{C}$, proving that f is constant. □

Corollary 4.2. *A holomorphic function defined on $\mathbb{C}$ into the upper-half plane $H = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ must be constant.*

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Proof. Note that a linear fractional transformation

$$F(z) = \frac{z-i}{z+i} \qquad (i = \sqrt{-1})$$

maps the upper-half plane H to the unit disc $D = \{w \in \mathbb{C} \mid |w| < 1\}$ bijectively. Then for each holomorphic function $f\colon \mathbb{C} \to H$, $F \circ f$ is a bounded holomorphic function defend on $\mathbb{C}$. □

Conformal minimal surfaces. Let $f\colon \Sigma \to \mathbb{R}^3$ be an immersion, where Σ is an orientable 2-dimensional manifold. As seen in Corollary 3.11, there exists a structure of Riemann surface such that each complex coordinate $z = u + iv$ gives an isothermal coordinate system.

Definition 4.3. An immersion $f\colon \Sigma \to \mathbb{R}^3$ of a Riemann surface Σ is said to be *conformal* if each complex coordinate $z = u + iv$ is isothermal.

In this section, we consider conformal minimal immersions $f\colon \Sigma \to \mathbb{R}^3$. Then by virtue of Proposition , and Lemma 3.4,

$$\phi := \frac{\partial f}{\partial z}\left(= \frac{1}{2}\left(\frac{\partial f}{\partial u} - i\frac{\partial f}{\partial v}\right)\right) : \Sigma \to \mathbb{C}^3 \tag{4.1}$$

is holomorphic for each complex coordinate $z = u + iv$ of Σ. Moreover, we have

Proposition 4.4. *Let $f\colon \Sigma \to \mathbb{R}^3$ be a conformal minimal immersion. Then for each complex coordinate chart $(U; z = u+iv)$*

of Σ, ϕ *in* (4.1) *satisfies*

$$(\phi_1)^2 + (\phi_2)^2 + (\phi_3)^2 = 0, \tag{4.2}$$

$$|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 > 0, \tag{4.3}$$

where we write $\phi = (\phi_1, \phi_2, \phi_3)$.

Proof. Since $\phi = (1/2)(f_u - i f_v)$,

$$\begin{aligned}(\phi_1)^2+(\phi_2)^2 + (\phi_3)^2 = \phi\cdot\phi &= \frac{1}{4}\bigl(f_u\cdot f_u - f_v\cdot f_v - 2i f_u\cdot f_v\bigr)\\ &= \frac{1}{4}\bigl((E-G) - 2iF\bigr) = 0,\end{aligned}$$

where E, F and G are the components of the first fundamental form $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 = E(du^2 + dv^2)$. Then (4.2) follows. On the other hand,

$$\begin{aligned}|\phi_1|^2+|\phi_2|^2 + |\phi_3|^2 = \phi\cdot\bar\phi &= \frac{1}{4}\bigl(f_u\cdot f_u + f_v\cdot f_v\bigr)\\ &= \frac{1}{4}(E+G) = \frac{E}{2} > 0,\end{aligned}$$

proving (4.3). □

Bernstein's Theorem We prove the following global result of minimal surfaces:

Theorem 4.5 (Bernstein, 1915)**.** *Let* $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ *be a smooth function defined on the whole plane* $\mathbb{R}^2$*, and assume the graph of* φ *is minimal surface. Then* $\varphi(x,y)$ *is a linear function in* (x,y)*. In other words, the only entire minimal graphs are planes.*

Proof. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ be a solution of the minimal surface equation

$$(1+\varphi_y^2)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} + (1+\varphi_x^2)\varphi_{yy} = 0. \tag{4.4}$$

Then there exists functions ξ and η satisfying

$$d\xi = \left(1 + \frac{1+\varphi_x^2}{W}\right)dx + \frac{\varphi_x\varphi_y}{W}\,dy, \tag{4.5}$$

$$d\eta = \frac{\varphi_x\varphi_y}{W}\,dx + \left(1 + \frac{1+\varphi_y^2}{W}\right)dy, \tag{4.6}$$

where $W = \sqrt{1+\varphi_x^2+\varphi_y^2}$. Moreover, by Proposition 3.13, we know that the map

$$\mathbb{R}^2 \ni (x,y) \longmapsto (\xi,\eta) \in \mathbb{R}^2$$

is a diffeomorphism and

$$f\colon \mathbb{C} \ni \zeta := \xi + i\eta \longmapsto \bigl(x(\xi,\eta), y(\xi,\eta), \varphi(x(\xi,\eta), y(\xi,\eta))\bigr) \in \mathbb{R}^3,$$

is a conformal reparametrization of the graph of φ. We let ϕ as in (4.1):

$$\phi = (\phi_1,\phi_2,\phi_3) = \frac{\partial f}{\partial\zeta} = \left(\frac{\partial x}{\partial\zeta}, \frac{\partial y}{\partial\zeta}, \frac{\partial\varphi}{\partial\zeta}\right), \qquad (\zeta = \xi + i\eta).$$

Since

$$4\,\mathrm{Im}\left(\phi_1\overline{\phi_2}\right) = 4\,\mathrm{Im}\left(x_\zeta\overline{y_\zeta}\right) = \mathrm{Im}(x_\xi - ix_\eta)(y_\xi + iy_\eta)$$
$$= x_\xi y_\eta - y_\xi x_\eta = \det\begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix} = \det\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}^{-1}$$
$$= \left(1 + \frac{1+\varphi_x^2}{W}\right)\left(1 + \frac{1+\varphi_y^2}{W}\right) - \frac{\varphi_x^2\varphi_y^2}{W^2} > 0, \tag{4.7}$$

both ϕ_1 and ϕ_2 never vanish, and

$$\mathrm{Im}\,\frac{\phi_1}{\phi_2} = \frac{\mathrm{Im}\,\phi_1\overline{\phi_2}}{|\phi_2|^2} > 0.$$

Then we have a holomorphic map of $\mathbb{C}$ into the upper half plane

$$\frac{\phi_1}{\phi_2} \colon \mathbb{C} \longrightarrow H.$$

Hence by Liouville's Theorem 4.1, we conclude that

$$\phi_1 = a\phi_2, \qquad \text{that is} \qquad \frac{\partial x}{\partial \zeta} = a\frac{\partial y}{\partial \zeta} \qquad (a \in \mathbb{C}\setminus\{0\}). \tag{4.8}$$

Moreover, by (4.7), we have

$$\mathrm{Im}(\phi_1\overline{\phi_2}) = \mathrm{Im}(a|\phi_2|^2) > 0, \quad \text{that is,} \quad \mathrm{Im}\,a > 0. \tag{4.9}$$

By (4.8), and noticing x and y are real valued functions, we have

$$\frac{\partial x}{\partial \bar\zeta} = \overline{\frac{\partial x}{\partial \zeta}} = \overline{a\frac{\partial y}{\partial \zeta}} = \bar a\frac{\partial y}{\partial \bar\zeta}.$$

Then, if we set $w = x + iy$,

$$\frac{\partial w}{\partial \bar\zeta} = \frac{\partial x}{\partial \bar\zeta} + i\frac{\partial y}{\partial \bar\zeta} = (\bar a + i)\frac{\partial y}{\partial \bar\zeta}, \qquad \frac{\partial \bar w}{\partial \bar\zeta} = \frac{\partial x}{\partial \bar\zeta} - i\frac{\partial y}{\partial \bar\zeta} = (\bar a - i)\frac{\partial y}{\partial \bar\zeta}$$

hold. We set

$$q := q(\zeta) = (-\bar a + i)w + (\bar a + i)\bar w, \quad \left(w(\zeta) = x(\zeta) + iy(\zeta)\right). \tag{4.10}$$

Then we have

$$\frac{\partial q}{\partial \bar\zeta} = (-\bar a + i)(\bar a + i)\frac{\partial y}{\partial \bar\zeta} + (\bar a + i)(\bar a - i)\frac{\partial y}{\partial \bar\zeta} = 0,$$

that is, $\zeta \mapsto q$ is a holomorphic function. If we write $q = u + iv$ and $a = s + it$, we have

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -2t \\ 2 & -2s \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad (t = \mathrm{Im}\,a > 0). \tag{4.11}$$

that is, x and y are linear functions of u and v.

By holomorphicity of w, (u, v) is also an isothermal parameter of the surface. We set

$$\tilde\phi = (\tilde\phi_1, \tilde\phi_2, \tilde\phi_3) := \left(\frac{\partial x}{\partial w}, \frac{\partial y}{\partial w}, \frac{\partial z}{\partial w}\right).$$

Since x and y are linear functions of u and v, $\tilde\phi_1$ and $\tilde\phi_2$ are constants. On the other hand, since w is an isothermal (complex) parameter, (4.2) holds for $\tilde\phi$:

$$\tilde\phi_3^2 = -\tilde\phi_1^2 - \tilde\phi_2^2 = \text{constant}.$$

Therefore, the third coordinate z is also a liner function of u and v. Hence

$$z(u,v) = \varphi\bigl(x(u,v), y(u,v)\bigr)$$

is a liner function in (u,v). Thus, by (4.11), $\varphi(x,y)$ is a linear function. □

References

[4-1] Osserman, R., A SURVEY OF MINIMAL SURFACES, Dover Publ.

Exercises

Solve one of the following problems:

4-1[H] Let $f\colon \mathbb{C} \subset U \to \mathbb{R}^3$ be a conformal minimal immersion and set $\phi = (\phi_1, \phi_2, \phi_3)$ as (4.1). Show that

(1) the first fundamental form of f is expressed as

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \qquad \text{where } e^{2\sigma} = 2(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2),$$

(2) the unit normal vector field ν is expressed as

$$\nu = \frac{f_u \times f_v}{|f_u \times f_v|} = \frac{-i(\phi_2\overline{\phi_3} - \phi_3\overline{\phi_2}, \phi_3\overline{\phi_1} - \phi_1\overline{\phi_3}, \phi_1\overline{\phi_2} - \phi_2\overline{\phi_1})}{|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2},$$

(3) and the composition of $\nu\colon U \to S^2$ with the stereographic projection

$$\pi \circ S^2 \ni (\nu_1, \nu_2, \nu_3) \longmapsto \frac{1-\nu_3}{\nu_1 + i\nu_2} \in \mathbb{C} \cup \{\infty\}$$

is expressed as

$$\pi \circ \nu = \frac{\phi_3}{\phi_1 - i\phi_2},$$

here $z = u + iv$ is the complex coordinate of U. (Hint: $\phi_3^2 = -(\phi_1 + i\phi_2)(\phi_1 - i\phi_2)$.)

4-2[H] Find a non-trivial (non-linear) solution $\varphi(x,y)$ of the partial differential equation

$$(1 - \varphi_y^2)\varphi_{xx} + 2\varphi_x\varphi_y\varphi_{xy} + (1 - \varphi_x^2)\varphi_{yy} = 0,$$

which is defined on whole $\mathbb{R}^2$ (Hint: Try a similar method as in 2).