5 The Weierstrass representation

Complex Analysis again. A holomorphic function f on a domain $D \subset \mathbb{C}$ is said to be *having an isolated singularity* at p if there exists a neighborhood U_p of p such that $U_p \subset D$.

Fact 5.1 (The Laurent expansion). *For a holomorphic function f having an isolated singularity at p, there exists a positive number ε and complex numbers a_n $(n \in \mathbb{Z})$ such that*

$$(5.1)\quad f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-p)^n \qquad (D_{p,\varepsilon} := \{z\,;0<|z-p|<\varepsilon\}).$$

The convergence of the right-hand side is uniform on any compact subset of $D_{p,\varepsilon}$

Definition 5.2. The coefficient a_{-1} in (5.1) is called the *residue* of f at an isolated singularity p, and denoted by

$$\operatorname*{Res}_{z=p} f(z) := a_{-1}.$$

Definition 5.3. An isolated singularity p of holomorphic function f is a *pole* of (at most) order k if $a_{-m}=0$ holds in (5.1) for $m>k$. If $\{m\,;a_m \neq 0\}$ is unbounded, p is said to be an *essential singularity.*

Proposition 5.4. *If p is a pole of order at most k of a holomorphic function f, the residue is computed as*

$$\operatorname*{Res}_{z=p} f(z) = \frac{1}{(k-1)!}\lim_{z\to p}\frac{d^{k-1}}{dz^{k-1}}\bigl\{(z-p)^k f(z)\bigr\}.$$

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Holomorphic differential forms on Riemann surfaces. Let M^2 be a Riemann surface, i.e., a 1-dimensional complex manifold and let $\{(U_\alpha, z_\alpha)\}$ be a complex atlas of it. A notion of holomorphic function on M^2 is defined in a usual way: a function $f\colon M^2 \to \mathbb{C}$ is holomorphic if $f|_{U_\alpha}$ is holomorphic in z_α for each α. A *meromorphic function* on M^2 is a holomorphic function on $M^2 \setminus \Sigma$, where Σ is a discrete subset of M^2, such that each point $p \in \Sigma$ is at most a pole of $f|_{U_\alpha}$ for a chart U_α containing p. The order of a pole at $p \in \Sigma$ is defined as the order of $f|_{U_\alpha}$ at p, which does not depend on a choice of coordinates.

A form "$f(z)\,dz$" on a local complex chart (U,z) is called a *holomorhpic* 1*-form* if $f(z)$ is a holomorphic function in z. A holomorphic 1-form on M^2 is a collection of holomorhpic 1-forms $\{f_\alpha\,dz_\alpha\}$ satisfying the compatibility

$$f_\alpha(z_\alpha) = f_\beta(z_\beta(z_\alpha))\frac{dz_\alpha}{dz_\beta}.$$

The collection $\{f_\alpha\,dz_\alpha\}$ is a meromorphic 1-form if each f_α is meromorphic.

Definition 5.5. Let $\omega = \{f_\alpha\,dz_\alpha\}$ be a meromorphic 1-form on M^2 and p a pole of ω. The *residue* of ω at p is defined as

$$\operatorname*{Res}_p \omega := \operatorname*{Res}_{z_\alpha=p} f_\alpha(z_\alpha),$$

were (U_α, z_α) is a complex chart around p.

Remark 5.6. The definition of the residue does not depend on a choice of coordinate charts.

Let C be a curve in a coordinate neighborhood (U, z) of M^2, and $z = z(t)$ $(a \leqq t \leqq b)$ is a parametrization of it. Then the integration of a holomorphic 1-form $f(z)\,dz$ on U is defined as

$$\int_C f(z)\,dz = \int_a^b f\bigl(z(t)\bigr)\,\frac{dz(t)}{dt}\,dt. \tag{5.2}$$

Noticing that this definition does not depend on coordinate charts and parametrizations of C, one can define the line integral of a holomorphic 1-form ω on M^2 along a curve C. The following is a corollary of Cauchy's theorem of complex integrations:

Fact 5.7 (The residue principle)**.** *Let C is a closed curve of M^2 which bound a simply connected domain $D \subset M^2$, and ω be a meromorhpic 1-form on a neighborhood of $D \cup C$ which have the only pole $p \in D$. Then*

$$\int_C \omega = 2\pi i \operatorname*{Res}_p \omega.$$

The Weierstrass representation formula. Let M^2 be an orientable manifold and $f\colon M^2 \to \mathbb{R}^3$ be an immersion. By Corollary 3.11, there exists a structure of Riemann surface on M^2 such that any complex coordinate is isothermal. So, without loss of generality, we may assume that M^2 is a Riemann surface and f is a conformal immersion. Moreover, if f is minimal,

$$\phi := \frac{\partial f}{\partial z} \colon U \longrightarrow \mathbb{C}^3 \tag{5.3}$$

is a holomorphic map satisfying (4.2) and (4.3), cf., Proposition 4.4, where (U, z) is a complex coordinate chart.

Though ϕ depends on a choice of coordinate charts,

$$\hat{\phi} := \phi(z)\,dz \tag{5.4}$$

does not depend on coordinates. In fact, if one take another complex coordinate chart (V, w),

$$\frac{\partial f}{\partial w}\,dw = \frac{\partial f}{\partial z}\frac{dz}{dw}\,dw = \frac{\partial f}{\partial z}\,dz.$$

Proposition 5.8 (The Weierstrass representation)**.** *For a conformal minimal immersion $f\colon M^2 \to \mathbb{R}^3$ of a Riemann surface M^2, there exists a meromorphic function g and a holomorphic 1-form ω on M^2 such that, up to translations in $\mathbb{R}^3$,*

$$f(z) = \operatorname{Re}\int_{C_z} \bigl((1-g^2), i(1+g^2), 2g\bigr)\omega \tag{5.5}$$

holds, where C_z is a path on M^2 joining a base point z_0 and z.

Proof. Define $\phi = (\phi_1, \phi_2, \phi_3)$ as in (5.3). If $\phi_1 - i\phi_2$ is equivalently zero, $\phi_3 = 0$ because of (4.2). In this case, the surface is a horizontal plane, and $g = 0$, $\omega = a\,dz$ satisfy the conclusion. Otherwise, let $g := \frac{\phi_3}{\phi_1 - i\phi_2}$ and $\omega = \phi_1 - i\phi_2$.

Since g does not depend on a choice of complex charts, g is a meromorphic function on M^2. On the other hand, by (5.4) does not depend on coordinates, ω can be considered as a holomorphic 1-form on M^2. By (4.2), we have

$$\begin{aligned} 0 &= \hat{\phi}_1^2 + \hat{\phi}_2^2 + \hat{\phi}_3^2 = (\hat{\phi}_1 - i\hat{\phi}_2)(\hat{\phi}_1 + i\hat{\phi}_2) + \phi_3^2 \\ &= (\hat{\phi}_1 - i\hat{\phi}_2)\omega + g^2\omega^2, \end{aligned}$$

which implies $\hat\phi_1 - i\hat\phi_2 = -g^2\omega$, where $\hat\phi = \phi\,dz$. Hence we have $\hat\phi = \frac{1}{2}\big((1-g^2), i(1+g^2), 2g\big)\omega$. Equation 5.5 holds because

$$(5.6)\quad F(z) := \int_{C_z} \hat\phi, \quad \text{then} \quad \frac{\partial}{\partial z}\big(F(z) + \overline{F}(z)\big) = \hat\phi. \qquad \square$$

Corollary 5.9. *Let f be as in (5.5), the first fundamental form ds^2, the unit normal vector field ν, and the second fundamental form II are expressed as*

$$(5.7)\qquad ds^2 = (1+|g|^2)^2|\omega|^2,$$

$$(5.8)\qquad \nu = \frac{1}{1+|g|^2}\big(2\operatorname{Re} g, 2\operatorname{Im} g, |g|^2-1\big) = \pi^{-1}(g),$$

$$(5.9)\qquad II = -\omega\,dg - \overline{\omega\,dg},$$

where $\pi\colon S^2 \to \mathbb{C}\cup\{\infty\}$ is the stereographic projection.

Proof. Let $z = u + iv$ be a complex coordinate. Then by the proof of (4.3), $ds^2 = E(du^2+dv^2) = E\,dz\,d\bar z$, where $E = 2(|\phi_1|^2+|\phi_2|^2+|\phi_3|^2)$ proving the first assertion. The second assertion was the homework 4-1. The third assertion follows since the second fandamental form is expressed as

$$II = (f_{zz}\cdot\nu)\,dz^2 + 2(f_{z\bar z}\cdot\nu)\,dz\,d\bar z + (f_{\bar z\bar z}\cdot\nu)\,d\bar z^2. \qquad \square$$

As seen in Corollary 5.9, the meromorphic function $g\colon M^2 \to \mathbb{C}\cup\{\infty\}$ can be identified with ν via the streographic projection. So we call g the *Gauss map* of f.

The following is the converse assertion of Proposition 5.8.

Theorem 5.10 (The Weierstrass representation)**.** *Let M^2 be a simply connected Riemann surface, and let g and ω be a pair of a meromorphic function and a holomorphic 1-form on M^2 such that ds^2 in (5.7) is positive definite[4]. Then (5.5) gives a minimal immersion.*

Proof. The integration (5.6) does not depend on a choice of paths C_z, and then it gives a map $F\colon M^2 \to \mathbb{C}^3$. $\square$

Examples.

Example 5.11. Let $M^2 = \mathbb{C}$, $(g,\omega) = (z, dz)$. Then

$$\begin{aligned} f := &\operatorname{Re}\int \big(1-z^2, i(1+z^2), 2z\big)\,dz \\ &= \left(u - \frac{u^3}{3} + uv^2, -v - u^2v + \frac{v^3}{3}, u^2 - v^2\right) \end{aligned}$$

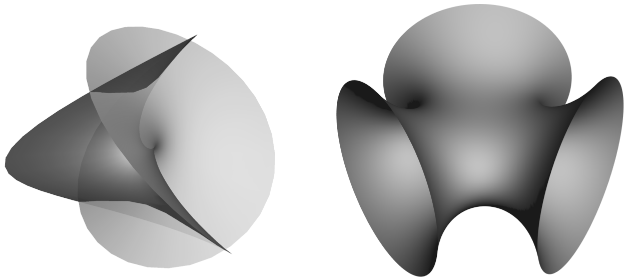
is a minimal surface, where $z = u + iv$ (Figure 3,left). This surface is known as *Enneper's surface.*

Example 5.12. Let $M^2 = \mathbb{C}\setminus\{0\}$ (not simply connected) and set $(g,\omega) = (z, i\,dz/z^2)$. Then f in (5.5) is represented by

$$f = \left(\left(r - \frac{1}{r}\right)\sin\theta, \left(r - \frac{1}{r}\right)\cos\theta, -2\theta\right) \qquad (z = re^{i\theta})$$

which is not well-defined on M^2 but on the universal cover of M^2. The surface is congruent to the helicoid (Example 5.12).

[4] This condition is equivalent that the set of the zeros of ω is the set of poles of g, and for each pole p of g, the order of the pole p of g is exactly half of the order of zero of ω.



Example 5.11 Example 5.16 ($n = 3$)

Figure 3: Examples of minimal surfaces.

Example 5.13. Let $M^2 = \mathbb{C} \setminus \{0\}$ and set $(g, \omega) = (z, dz/z^2)$. Then f in (5.5) is represented by, with $z = re^{i\theta}$,

$$f = \left(-\left(r + \frac{1}{r} \right) \cos\theta, -\left(r + \frac{1}{r} \right) \sin\theta, 2\log r \right) : M^2 \to \mathbb{R}^3,$$

which is the catenoid (Example 5.13).

The phenomenon as in Example 5.13 is generalized as

Proposition 5.14. *Let M^2 be a (not necessarily simply connected) Riemann surface, and let (g, ω) be a pair of a meromorphic function and a holomorphic 1-form on M^2 such that ds^2 in (5.7) is positive definite. Assume*

$$\operatorname{Re} \int_\gamma \bigl(1 - g^2, i(1 + g^2), 2g\bigr)\omega = 0$$

holds for all loops γ on M^2. Then f in (5.5) is well-defined on M^2 and gives a minimal immersion of M^2 into $\mathbb{R}^3$.

Corollary 5.15. *Let $M^2 = \mathbb{C} \cup \{\infty\} \setminus \{p_1, \dots, p_n\}$, and (g, ω) as in Proposition 5.14, and assume*

$$\operatorname{Im} \operatorname*{Res}_{p_j} \bigl(1 - g^2, i(1 + g^2), 2g\bigr)\omega = 0 \qquad (j = 1, \dots, n).$$

Then f as in (5.5) is a minimal immersion defined on M^2.

Example 5.16. Let $n \geqq 2$ be an integer, and

$$M^2 = \mathbb{C} \cup \{\infty\} \setminus \{1, \zeta, \dots, \zeta^{n-1}\}, \qquad \zeta = e^{2\pi i/n}.$$

Then $(g, omega) = \left(z^{n-1}, \frac{dz}{(z^n - 1)^2} \right)$ satisfies the assumptions of Corollary 5.15, and hence there exists a minimal immersion $f\colon M^2 \to \mathbb{R}^3$ with (g, ω). Such a series of minimal surfaces are called the *Jorge-Meeks' surfaces.*

References

[5-1] R. Osserman, A SURVEY OF MINIMAL SURFACES, Dover Publ.

[5-2] L. P. Jorge and W. H. Meeks, III, *The topology of complete minimal surfaces of finite total Gaussian curvature*, Topology **22** (1983), 203–221.

Exercises

5-1[H] Verify Example 5.16 for $n = 3$.