6 Further Example

Completeness and finiteness of topology It is well-known that that there exist no compact minimal surfaces without boundaries. So to investigate global properties of minimal surfaces, we need a notion of completeness as follows: A Riemannian 2-manifold (M^2, ds^2) is said to be *complete* if all divergent paths have finite length. Here, a path $\gamma\colon [0,\infty) \to M^2$ is *divergent*, if, for each compact set $K \subset M^2$, there exists a positive number m such that $\gamma\bigl([m,+\infty)\bigr) \subset M^2 \setminus K$.

One can check that the plane, the catenoid (Examples 5.13 and 2.4), the helicoid (Examples 5.12 and 2.6) (and Examples in Sections 2 and 5) are complete.

The following result is known (Osserman [6-3]):

Fact 6.1 (Osserman, 1961)**.** *Let $f\colon M^2 \to \mathbb{R}^3$ be a complete minimal immersion of an orientable manifold M^2 with finite total curvature. Then there exists a compact Riemann surface $\overline{M}^2$ and finite number of points $\{p_1,\dots,p_n\}$ such that M^2 (with complex structure induced by the first fundamental form) is biholomorphic to $\overline{M}^2 \setminus \{p_1,\dots,p_n\}$.*

Here, the total curvature of the minimal surface $f\colon M^2 \to \mathbb{R}^3$ is the integral of the Gaussian curvature K: $\mathrm{TC}(f) := \int_{M^2} K\,dA$. Since K is non-negative for minimal surfaces, $\mathrm{TC}(f)$ is valued on $[-\infty, 0]$.

Scherk's surface (Example 2.2; extended to the doubly periodic surface), and the helicoid are complete but not of finite total curvature. On the other hand, the total curvature of the catenoid is -4π, which is finite. Moreover, the Jorge-Meeks surface (Example 5.16) has total curvature $-4(n-1)\pi$.

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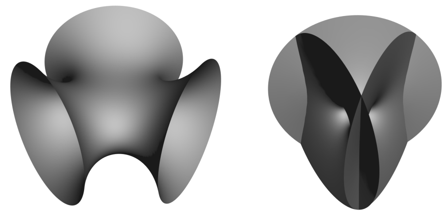

Figure 4: The Jorge-Meeks surface for $n = 3$.
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Embeddedness of minimal surfaces is also important global property. Scherk's surface, the catenoid and the helicoid are embedded, but the Jorge-Meeks surfaces for $n \geqq 3$ are not-embedded (Figure 4).

Costa's example In this section, we introduce an example of compact embedded minimal surface with finite total curvature, firstly discovered by Costa [6-1].

Domain and the Weierstrass data Take a holomorphic function of two variables $F(z,w) := w^2 - z(z^2-1)$ and set

$$(6.1)\quad M_0 := \{(z,w) \in \mathbb{C}^2\,;\, w^2 = z(z^2-1)\} = F^{-1}(\{(0,0)\}).$$

Since $(F_z, F_w) \neq (0,0)$, M_0 is a complex submanifold of $\mathbb{C}^2$, by the (complex) implicit function theorem, and it is homeomorphic to a torus with one point excluded. The functions z and w are holomorphic on M_0. Since for each $z \neq 0, \pm 1$, there exists exactly two w's satisfying $F(z,w) = 0$, M_0 is a branched double cover of the Riemann sphere $\mathbb{C} \cup \{\infty\}$.[5] We set

$$M^2 := M_0 \setminus \{(\pm 1, 0)\}, \qquad g := \frac{\alpha}{w}, \qquad \omega := \frac{z\,dz}{w}, \tag{6.2}$$

where α is a positive constant defined later. Then one can easily check that (4.3) for ϕ holds on M^2. We prove the following

Proposition 6.2 (Costa)**.** *The Weierstrass data (g, ω) induces a minimal immersion of M^2 into $\mathbb{R}^2$.*

To show this, it is sufficient to show that

$$\int_\gamma \hat{\phi} \in i\mathbb{R}^3 \tag{6.3}$$

holds for all loops γ on M_0, where (cf. Proposition 5.14).

$$\hat{\phi} = (\hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3) := \bigl(1 - g^2, i(1+g^2), 2g\bigr)\omega.$$

Moreover, by Cauchy's theorem on complex integration, we only have to show (6.3) for generators of the fundamental group of M_0. Let $\beta_{\pm 1}$, β_∞ and γ_1 and γ_2 be loops as in Figure 5. Then these loops generates the fundamental group of M^2. We shall prove (6.3) for these loops.

[5]Such a double cover of the sphere is called a *hyperelliptic curve*.

Figure 5:

Remark that since $w^2 = z(z^2-1)$ holds on M_0, we have

$$\hat{\phi}_3 = 2g\omega = \frac{2\alpha\, z\, dz}{w^2} = \frac{2\alpha\, dz}{z^2 - 1} = d\left(\alpha \log \frac{z-1}{z+1}\right),$$

and so

$$\operatorname{Re} \int \hat{\phi}_3 = \alpha \log \left| \frac{z-1}{z+1} \right|,$$

which is well-defined on M^2, that is, (6.3) holds for an arbitrary loop γ. Thus, we only consider the periods for $\hat{\phi}_1$ and $\hat{\phi}_2$.

The period about $\beta_{\pm 1}$. Since $F_z(\pm 1, 0) \neq 0$, w is a local complex coordinate near $(\pm 1, 0)$, and

$$dF = 2w\,dw - (3z^2 - 1)\,dz = 0$$

holds on M_0. Thus, we have

$$\omega = \frac{z\,dz}{w} = \frac{2z\,dw}{3z^2 - 1},$$

that is, ω is holomorphic at $(\pm 1, 0)$. On the other hand,

$$g^2\omega = \frac{2\alpha^2\, z\, dz}{w^3} = \frac{2\alpha^2\, z}{3z^2-1}\frac{dw}{w^2},$$

that is, $g^2\omega$ has a pole of order 2 at $(z,w) = (\pm 1, 0)$. Moreover,

$$\frac{d}{dw}\frac{z}{3z^2-1} = \frac{dz}{dw}\frac{d}{dz}\frac{z}{3z^2-1} = \frac{2w}{3z^2-1}\frac{-2(3z^2+1)}{(3z^2-2)}.$$

Hence the residue of $g^2\omega$ at $(z,w) = (\pm 1, 0)$ vanishes.

Hence the integrals of $\hat{\phi}_1$ and $\hat{\phi}_2$ along the loops $\beta_{\pm 1}$ vanish.

The period about β_∞. The loop β_∞ is considered as a loop surrounding $(z,w) = (\infty, \infty)$. We set $u = 1/z$, $v = 1/w$. Then the equation $F(z,w) = 0$ is equivalent to

$$G(u,v) := u^3 - v^2(1-u^2) = 0.$$

Unfortunately, the derivatives of G vanish at $(0,0)$. So we take (regularized) coordinate system (u,s) such that $v = su$ (this procedure is known as *blowing-up*). Then

$$\tilde{G}(u,s) = u - s^2(1-u^2) = 0$$

corresponds to the defining equation of M_0. Using these coordinates, the Weierstrass data can be rewritten as

$$g = \frac{\alpha u}{s}, \qquad \omega = \frac{z\, dz}{w} = \frac{u}{s}\frac{1}{u}d\left(\frac{1}{u}\right) = -\frac{s\, du}{u^2}.$$

Hence by the relations

$$s = \frac{u}{1-u^2}, \quad 2s\, ds = d\left(\frac{u}{1-u^2}\right) = \frac{u^2+1}{(u^2-1)^2}\, du,$$

we have

$$\omega = \frac{-2\, ds}{s^2(1+u^2)}, \qquad g^2\omega = \frac{-2\alpha^2\, ds}{(1+u^2)(1-u^2)^2}.$$

Hence $g^2\omega$ is holomorphic and ω has a pole of order 2 at $(0,0)$. By the similar way as the case at $z = \pm 1$, we can compute that the residue of ω vanish at $s = 0$. Hence the integrals of $\hat{\phi}_j$ $(j = 1, 2)$ along β_∞ vanish.

The period along γ_1. Consider the loop $\gamma_1 = \gamma_1^+ \cup \gamma_1^-$. Since ω is holomorphic on a neighborhood of γ_1, its integral along $\gamma_1^\pm$ reduces to

$$\int_{\gamma_1^\pm} = \int_0^1 \frac{dt}{\sqrt{t(1-t^2)}} =: A > 0 \tag{6.4}$$

On the other hand, $g^2\omega$ has a pole of order 2 at $(z,w) = (-1,0)$, and the integration along the interval $[-1,0]$ diverges. So we consider a loop $z = -1/2 + re^{i\theta}$ on the z-plane. Then

$$\int_{\gamma_1^+} g^2\omega = \alpha^2 B_r, \qquad \int_{\gamma_1^-} g^2\omega = \alpha^2\overline{B_r},$$

where

$$B_r := \int_0^\pi \frac{ire^{i\theta}\, d\theta}{(z^2-1)\sqrt{z(z-1)(z+1)}} \qquad \left(z = -\frac{1}{2} + re^{i\theta}\right).$$

Since integration along γ_1 does not depend on a choice of $r \in (\frac{1}{2}, 1)$, we can set

$$(6.5)\quad B := \operatorname{Re} B_r = \operatorname{Re} \int_0^{\pi} \frac{ie^{i\theta}\,d\theta}{\sqrt{z}\sqrt{z^2-1}^3} \quad \left(z = -\frac{1}{2} + e^{i\theta}\right).$$

Moreover, one can check that $B > 0$ hols. Hence, if we choose $\alpha = \sqrt{A/B}$, we have

$$\int_{\gamma_1} \hat{\phi}_1 = \int_{\gamma_1} (1-g^2)\omega = 2(A - \alpha^2 B) = 0.$$

On the other hand,

$$\int_{\gamma_1} \hat{\phi}_2 = \int_{\gamma_1} i(1+g^2)\omega = 2i(A + \alpha^2 B) \in i\mathbb{R}.$$

The period along γ_2. Consider a map $\mathbb{C}^2 \ni (z, w) \mapsto (-z, iw) \in \mathbb{C}^2$, which induces an automorphism of the torus M_0. This morphism maps the loop γ_1 to γ_2, and $(g, \omega) \mapsto (-ig, -i\omega)$. Thus, we have

$$\int_{\gamma_2} \hat{\phi}_1 = -2i(A + \alpha^2 B), \quad \int_{\gamma_2} \hat{\phi}_2 = 2(A - \alpha^2 B) = 0.$$

Hence (6.3) is accomplished, and Proposition **??** is proven.

The minimal surface we have obtained such a way is called *Costa's surface*. Costa's surface is a complete, embedded minimal surface of genus 1, with 3 ends, whose total curvature is -12π.

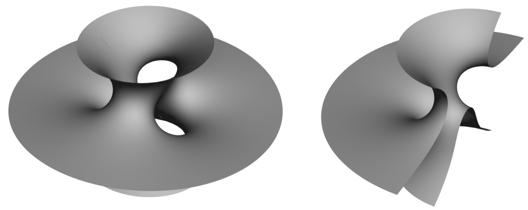

Figure 6: Costa's surface

References

[6-1] C. J. Costa, *Example of a complete minimal immersion in $\mathbb{R}^3$ of genus one and three embedded ends*, Bol. Soc. Brasil. Mat. **15** (1984), 47–54.

[6-2] D. Hoffman and W. MeeksIII, *Embedded minimal surfaces of finite topology*. Ann. of Math. (2) **131** (1990), 1–34.

[6-3] R. Osserman, A SURVEY OF MINIMAL SURFACES, Dover Publ.

Exercises

6-1$^{\mathrm{H}}$ Show that the third coordinate of Costa's surface is bounded as $(z, w) \to (\infty, \infty)$.