Surfaces of constant mean curvature

Closed surfaces A *closed surface* in the Euclidean 3-space $\mathbb{R}^3$ is a C^∞-immersion $f\colon \Sigma \to \mathbb{R}^3$ of a compact, connected 2 dimensional manifold Σ into $\mathbb{R}^3$. Taking a local coordinate neighborhood $(U; u, v)$ of Σ, f can be identified a parametrized surface $f(u,v)$ as in the previous section.

Throughout this section, we assume that Σ is *oriented*, that is, an atlas $\{(U_\alpha; u^\alpha, v^\alpha) \mid \alpha \in A\}$ of Σ satisfying

$$\frac{\partial(u^\beta, v^\beta)}{\partial(u^\alpha, v^\alpha)} := \det J_{\alpha\beta} > 0 \qquad \text{on } U_\alpha \cap U_\beta \tag{2.1}$$

for each α, $\beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$ is specified. Here $J_{\alpha\beta}$ is the *Jacobian matrix* of the coordinate change $(u^\alpha, v^\alpha) \mapsto (u^\beta, v^\beta)$

$$J_{\alpha\beta} := \begin{pmatrix} \frac{\partial u^\beta}{\partial u^\alpha} & \frac{\partial u^\beta}{\partial v^\alpha} \\ \frac{\partial v^\beta}{\partial u^\alpha} & \frac{\partial v^\beta}{\partial v^\alpha} \end{pmatrix} \tag{2.2}$$

Fix a coordinate neighborhood $(U; u, v)$. Then the immersion $f\colon (u,v) \mapsto f(u,v)$ is considered as a vector-valued smooth function on U, and so are there derivatives f_u and f_v. Then the unit normal vector ν, the first fundamental form ds^2, the second fundamental form $I\!I$, the area element dA, the Gaussian curvature K and the mean curvature H are defined as in (1.1), (1.2), (5.3) and (1.5) in the previous section. Moreover, one can prove easily that they are independent on choice of local coordinate systems (cf. [2-1] and/or [2-2]).

17. April, 2018. Revised: 24. April, 2018

Definition 2.1. Let $f\colon \Sigma \to \mathbb{R}^3$ be an oriented closed surface. Then the *area* $\mathcal{A}(f)$ of $f(\Sigma)$ and the (signed) *volume* $\mathcal{V}(f)$ of the region bounded by $f(\Sigma)$ are defined as

$$\mathcal{A}(f) := \int_\Sigma dA, \qquad \mathcal{V}(f) := \frac{1}{3}\int_\Sigma f \cdot \nu \, dA,$$

where "$\cdot$" denotes the canonical inner product of $\mathbb{R}^3$, ν is the unit normal vector as in (1.1), and dA denotes the area element which is represented by $dA := |f_u \times f_v| \, du \, dv$ on each coordinate neighborhood $(U; u, v)$.

Remark 2.2. If the surface f is an embedding, that is, the map f is injective (in this case), the image $f(\Sigma)$ bounds a bounded and connected region D of $\mathbb{R}^3$, and the volume of D coincide with the absolute value of $\mathcal{V}(f)$.

Obviously, these two functionals have the following properties:

Lemma 2.3. *For an immersion $f \in \mathcal{S}(\Sigma)$ and a positive number $\lambda > 0$, $\mathcal{A}(\lambda f) = \lambda^2 \mathcal{A}(f)$, and $\mathcal{V}(\lambda f) = \lambda^3 \mathcal{V}(f)$ hold.*

Example 2.4 (The round sphere)**.** Let $R > 0$ be a constant and denote by

$$S^2(R) := \{\boldsymbol{x} \in \mathbb{R}^3 \mid |\boldsymbol{x}| = R\} \subset \mathbb{R}^3$$

the sphere in $\mathbb{R}^3$ of radius R centered at the origin. Then the inclusion map

$$\iota\colon S^2(R) \ni \boldsymbol{x} \longmapsto \iota(\boldsymbol{x}) = \boldsymbol{x} \in \mathbb{R}^3$$

is an embedding. A map

$$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\times(-\pi,\pi)\ni(u,v)$$
$$\longmapsto (R\cos u\cos v, R\cos u\sin v, R\sin u)\in S^2(R)$$

gives a local coordinate system of $S^2(R)$, and we have

$$dA = R^2\cos u\,du\,dv,\qquad \nu = -(\cos u\cos v,\cos u\sin v,\sin u).$$

Since this coordinate neighborhood covers an open dense subset of $S^2(R)$, "integration over $S^2(R)$" is replaced by "integration over $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[-\pi,\pi]$":

$$\begin{aligned}\mathcal{A}(\iota) &= \int_{-\pi/2}^{\pi/2} du\int_{-\pi}^{\pi} dv\,R^2\cos u\\ &= 2\pi R^2\int_{-\pi/2}^{\pi/2}\cos u\,du = 4\pi R^2,\\ \mathcal{V}(\iota) &= \frac{1}{3}\int_{-\pi/2}^{\pi/2}\int_{-\pi}^{\pi} R^3\cos u\,du\,dv = -\frac{4}{3}\pi R^3.\end{aligned}$$

The Gaussian and the mean curvature are computed as

$$K=\frac{1}{R^2}\qquad\text{and}\qquad H=\frac{1}{R},$$

respectively, which are constant on the surface. We call S^2_R the *round sphere* of radius R.

Area minimizing surfaces with a volume constraint. Let Σ be a compact, connected and oriented 2-manifold and consider

$$\mathcal{S}(\Sigma)=\{f\colon\Sigma\to\mathbb{R}^3\mid f\text{ is an immersion}\}. \tag{2.3}$$

In addition, for a fixed positive constant V_0. we set

$$\mathcal{S}(\Sigma,V_0):=\{f\in\mathcal{S}(\Sigma)\mid\mathcal{V}(f)=V_0\}, \tag{2.4}$$

that is, $\mathcal{S}(\Sigma,V_0)$ is the set of immersions of Σ into $\mathbb{R}^3$ bounding given volume V_0.

In this section, we shall prove

Theorem 2.5. *If $f_0\in\mathcal{S}(\Sigma,V_0)$ minimizes the area in $\mathcal{S}(\Sigma,V_0)$, the mean curvature of f_0 is non-zero constant.*

Theorem 2.5 and Example 2.4 give rise to the following question, known as Heinz-Hopf's problem:

Question 2.6. *Are there a closed surface of constant mean curvature which is not congruent to the round sphere?*

Variation formula for the area and the volume Similar to the previous section, we define variations of $f\in\mathcal{S}(\Sigma)$:

Definition 2.7. A *variation* of an immersion $f\colon\Sigma\to\mathbb{R}^3$ is a C^∞-map $F\colon(-\varepsilon,\varepsilon)\times\Sigma\to\mathbb{R}^3$ satisfying

- $f^t:=F(t,*)\colon\Sigma\to\mathbb{R}^3$ is an immersion for each $t\in(-\varepsilon,\varepsilon)$,
- $f^0=F(0,*)$ coincides with f.

The variational vector field V of a variation $F = \{f^t\}$ is a vector-valued function V on Σ defined by

$$V(p) := \left.\frac{\partial}{\partial t}\right|_{t=0} F(t,p) \qquad (p \in \Sigma).$$

Similar to variational formula in Section 1, we have

Theorem 2.8. *Let $\{f^t\}$ be a variation of an immersion $f\colon \Sigma \to \mathbb{R}^3$. Then*

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(f^t) = -2\int_\Sigma H\varphi\, dA, \qquad \left.\frac{d}{dt}\right|_{t=0} \mathcal{V}(f^t) = \int_\Sigma \varphi\, dA,$$

hold, where $\varphi := V \cdot \nu$, V is the variational vector field of $\{f^t\}$ and ν is the unit normal vector field of f.

Proof. Since almost all part of the computation in the previous section are coordinate-independent, we can show the result in a similar way to them.

Here, we shall prove the formula for the volume functional. Let $(U; u, v)$ be a local coordinate system. Then it holds that

$$\begin{aligned}\varPhi := f^t \cdot \nu^t |f^t_u \times f^t_v| &= f^t \cdot \frac{f^t_u \times f^t_v}{|f^t_u \times f^t_v|}\,|f^t_u \times f^t_v| \\ &= \det(f^t, f^t_u, f^t_v)\end{aligned}$$

Differentiating this in t, we have

$$\begin{aligned}\left.\frac{\partial}{\partial t}\right|_{t=0} \varPhi &= \det(\dot f^t, f_u, f_v) + \det(f, \dot f^t_u, f_v) + \det(f, f_u, \dot f^t_v) \\ &= \det(V, f_u, f_v) + \det(f, V_u, f_v) + \det(f, f_u, V_v),\end{aligned}$$

where $\dot{*} = (\partial/\partial t)|_{t=0}$. Here, since

$$\begin{aligned}\det(V, f_u, f_v) &= V \cdot (f_u \times f_v) = (V \cdot \nu)|f_u \times f_v|, \\ \det(f, V_u, f_v) &= \big(\det(f, V, f_v)\big)_u - \det(f, V, f_{uv}) - \det(f_u, V, f_v) \\ &= \big(\det(f, V, f_v)\big)_u - \det(f, V, f_{uv}) + \det(V, f_u, f_v) \\ \det(f, f_u, V_v) &= \big(\det(f, f_u, V)\big)_v - \det(f, f_{uv}, V) - \det(f_v, f_u, V) \\ &= \big(\det(f, f_u, V)\big)_v - \det(f, f_{uv}, V) + \det(V, f_u, f_v),\end{aligned}$$

it holds that

$$\begin{aligned}\left(\left.\frac{\partial}{\partial t}\right|_{t=0} \varPhi\right) du \wedge dv &= 3(V \cdot \nu)\,|f_u \times f_v|\, du \wedge dv \\ &\quad + \Big(\big(\det(f, V, f_v)\big)_u + \big(\det(f, f_u, V)\big)_v\Big)\, du \wedge dv.\end{aligned}$$

Here, setting

$$\alpha := \det(f, V, f_u)\, du + \det(f, V, f_v)\, dv = \det(f, V, df),$$

we have the coordinate-independent expression

$$\left(\left.\frac{\partial}{\partial t}\right|_{t=0} \varPhi\right) du \wedge dv = 3(V \cdot \nu)\, dA + d\alpha,$$

and then,

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \mathcal{V}(f^t) &= \frac{1}{3}\int_\Sigma \left(\left.\frac{\partial}{\partial t}\right|_{t=0} \varPhi\right) du \wedge dv \\ &= \int_\Sigma (V \cdot \nu)\, dA + \frac{1}{3} d\alpha = \int_\Sigma (V \cdot \nu)\, dA,\end{aligned}$$

proving the formula. □

Proof of Theorem 2.5. Let $f_0 \in \mathcal{S}(\Sigma, V_0)$ be an immersion minimizing area in $\mathcal{S}(\Sigma, V_0)$. Then it holds that

$$(2.5)\qquad \left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(f^t) = 0 \quad \text{for any volume preserving variation } \{f^t\}.$$

Here, a variation $\{f^t\}$ of f_0 is said to be *volume preserving* if $\mathcal{V}(f^t) = \mathcal{V}(f_0)$ for all t.

Let $\{f^t\}$ be a (not necessarily volume preserving) variation of f_0. Then, by Lemma 2.3, $\{\tilde{f}^t\}$ defined by

$$\tilde{f}^t := \frac{\mathcal{V}(f^t)^{-1/3}}{\mathcal{V}(f_0)^{-1/3}}\, f^t$$

is volume preserving variation, and the map $\{f^t\} \mapsto \{\tilde{f}^t\}$ is a surjection to the set of volume preserving variations. That is, (2.5) is equivalent to

$$(2.6)\qquad \left.\frac{d}{dt}\right|_{t=0} \mathcal{A}\left(\frac{\mathcal{V}(f^t)^{-1/3}}{\mathcal{V}(f_0)^{-1/3}}\, f^t\right) = 0 \quad \text{for any variation } \{f^t\}.$$

Here, by Theorem 2.8,

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \mathcal{A}\big(\mathcal{V}(f^t)^{-1/3} f^t\big) &= \left.\frac{d}{dt}\right|_{t=0} \mathcal{V}(f^t)^{-2/3}\mathcal{A}(f^t) \\
&= -\frac{2}{3}\dot{\mathcal{V}}(f^t)\mathcal{V}(f_0)^{-5/3}\mathcal{A}(f_0) + \mathcal{V}(f_0)^{-2/3}\dot{\mathcal{A}}(f^t) \\
&= \mathcal{V}(f_0)^{-2/3}\left(-\frac{2}{3}\frac{\mathcal{A}(f_0)}{\mathcal{V}(f_0)}\dot{\mathcal{V}}(f^t) + \dot{\mathcal{A}}(f^t)\right) \\
&= \mathcal{V}(f_0)^{-2/3}\left(\int_\Sigma \left(-\frac{2}{3}\frac{\mathcal{A}(f_0)}{\mathcal{V}(f_0)} - 2H\right)\varphi\, dA\right),
\end{aligned}$$

where $\dot{*} = (d/dt)|_{t=0}$ and $\varphi = V \cdot \nu$. Then by Lemma 1.7,

$$-\frac{2}{3}\frac{\mathcal{A}(f_0)}{\mathcal{V}(f_0)} - 2H = 0,$$

holds, and then H is constant.

References

[2-1] 梅原雅顕，山田光太郎，曲線と曲面（改訂版），裳華房，2014.

[2-2] Masaaki Umehara and Kotaro Yamada, Differential Geometry of Curves and Surfaces, (trasl. by Wayne Rossman), World Scientific, 2017.

Exercises

2-1[H] Let $\mathcal{C} := \{\gamma\colon S^1 \to \mathbb{R}^2 \mid \gamma' \neq \mathbf{0}\}$ be the set of regular closed curves on $\mathbb{R}^2$.

(1) Define the area $\mathcal{A}(\gamma)$ of the region bounded by γ.

(2) Let $\mathcal{C}(a)$ be the set of curves γ with $\mathcal{A}(\gamma) = a$. Show that if a curve $\gamma_0 \in \mathcal{C}(a)$ minimizes the length in $\mathcal{C}(a)$, the curvature of γ_0 is constant.

Hint: A curve $\gamma \in \mathcal{C}(a)$ can be parametrized $\gamma(t) = {}^t(x(t), y(t))$ as a 2π-periodic function. The length $\mathcal{L}(\gamma)$ and the curvature function κ of γ are defined as

$$\mathcal{L}(\gamma) := \int_0^{2\pi} |\dot{\gamma}(t)|\, dt, \qquad \kappa(t) := \frac{\det(\dot{\gamma}(t), \ddot{\gamma}(t))}{|\dot{\gamma}(t)|^3}$$

where $\dot{} = d/dt$.