Delaunay Surfaces

Constant mean curvature surfaces of revolution. As seen in Theorem 3.6 in the previous section, we have

Theorem 4.1. *Let H and a be arbitrary constants with*

$$H > 0 \qquad \text{and} \qquad 2Ha + 1 > 0, \tag{4.1}$$

and let $\gamma(s) = \bigl(x(s), y(s)\bigr)$ with

$$\begin{aligned} y(s) &= \frac{1}{2|H|}\sqrt{(2Ha+1)^2 - 2(2Ha+1)\cos 2Hs + 1}, \\ x(s) &= \int_0^s \frac{(2Ha+1)\cos 2Hu - 1}{2Hy(u)}\,du. \end{aligned} \tag{4.2}$$

Then the surface of revolution with respect to the x axis with profile curve γ has constant mean curvature H. Conversely, constant mean curvature surfaces of revolution are obtained in this manner.

Proof. Take $H \neq 0$ and a arbitrarily. Then one can easily show that the surface of revolution with profile curve $(x(s), y(s))$ has constant mean curvature H.

On the other hand, in the proof of Theorem 3.6, we have solved the differential equation (3.8) with initial condition

$$\bigl(x(0), y(0)\bigr) = (0, a), \quad \bigl(\dot{x}(0), \dot{y}(0)\bigr) = (1, 0), \quad \ddot{y}(0) = \kappa(0) \leqq 0,$$

where a is a positive constant. In this case, (3.6) yields that

$$2H = \kappa - \frac{\dot{x}}{y} < 0,$$

and

$$2Ha + 1 = a\kappa(0) \leqq 0.$$

Hence, an arbitrary (non-zero) constant mean curvature surface of revolution is obtained by one of the expression (4.2) for $H < 0$ and $2Ha + 1 \leqq 0$.

Here, a surface obtained by replacing (H, a, s) by $(-H, -a, -s)$ in the expression (4.2) is congruent to the original one. Hence we may assume $H > 0$ without loss of generality. In addition, the change

$$(H, a, s) \mapsto (H', a', s') = \left(H, -a - \frac{1}{H}, s + \frac{\pi}{2H}\right)$$

keeps the curve unchanged, and $2H'a' + 1 = -(2Ha+1)$. Thus, we may assume $2Ha + 1 \geqq 0$ without loss of generality. □

Special Cases. Let $H > 0$ be a positive constant.

Example 4.2 (The circular cylinder)**.** When $a = -1/(2H)$, (4.2) turns to be

$$x(s) = -s, \qquad y(s) = \frac{1}{2H}.$$

The corresponding surface is a circular cylinder of radius $1/(2H)$.

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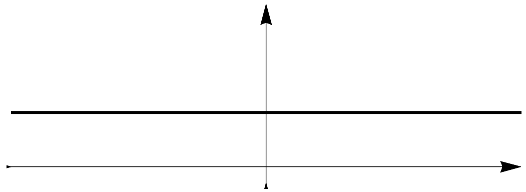

Figure 1: Example 4.2

Example 4.3 (The spheres)**.** When $a = 0$, (4.2) turns to be

$$\dot{x}(s) = -|\sin Hs|, \qquad y(s) = \frac{|\sin Hs|}{H}.$$

Integrating the first equation with respect to s, we have

$$x(s) = \frac{1}{H}\cos H\tau - \frac{n+1}{H}, \quad (Hs = n\pi + \tau, n \in \mathbb{Z}, \tau \in [0, \pi)).$$

The corresponding surface has singularities on $s = n\pi$ ($n \in \mathbb{Z}$), and its image is a sequence of infinitely many spheres with radius $1/H$ centered at $\frac{n\pi}{2H}$ ($n \in \mathbb{Z}$).

Generic Cases: Unduloids and Nodoids If $2Ha + 1 \notin \{0, 1\}$, $y(s)$ in (4.2) defined on $\mathbb{R}$ because

$$\begin{aligned}(2Ha+1)^2 &- 2(2Ha+1)\cos 2Hs + 1 \\ &\geqq (2Ha+1)^2 - 2(2Ha+1) + 1 = (2Ha)^2 > 0,\end{aligned}$$

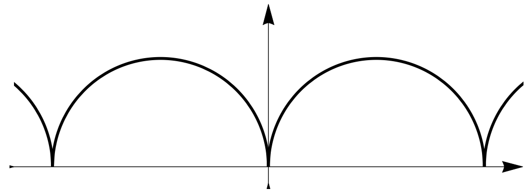

Figure 2: Example 4.3

and obviously $2\pi/H$-periodic. On the other hand, $x(s)$ in (4.2) has the following periodicity:

$$\begin{aligned} x\left(s + \frac{2\pi}{H}\right) &= x(s) + c, \\ c &:= \int_0^{2\pi/H} \frac{(2Ha+1)\cos 2Hu - 1}{2Hy(u)}\,du. \end{aligned} \tag{4.3}$$

Remark that the integration in (4.3) cannot be expressed in terms of elementary functions. In fact, it is an elliptic integral.

Proposition 4.4. *Let $H > 0$ and $a \in (-1/(2H), 0)$ be constants. Then $x(s)$ in (4.2) is a decreasing function Sikh $\dot{x}(s) < 0$. Then the curve $\gamma(s) = (x(s), y(s))$ has no self-intersection, and can be expressed as the graph $y = f(x)$.*

Proof. Since $0 < 2Ha + 1 < 1$, $(2Ha+1)\cos 2Hs - 1 < 0$ for each s. Hence

$$\dot{x}(s) = \frac{(2Ha+1)\cos 2Hs - 1}{2Hy(u)} < 0,$$

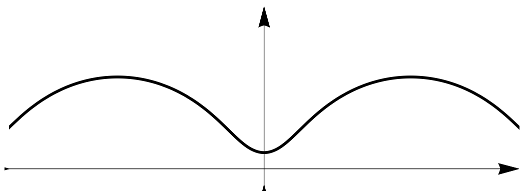

Figure 3: an unduloid

and the conclusion follows. □

A surface as in Proposition 4.4 is called an *unduloid* (Figure 3).

On the other hand, when $a > 0$, $x(s)$ is neither increasing nor decreasing. In fact,

Proposition 4.5. *Let $H > 0$ and $a > 0$ be constants. Then the curve $\gamma(s) = (x(s), y(s))$ in (4.2) have countably many self-intersections.*

A surface as in Proposition 4.5 is called an *nodoid* (Figure 4).

Plotting Delaunay surfaces. The profile curve of an unduloid is obtained as the locus of a focal point of an ellipse while rolling it without slippage along a given line (C. E. Delaunay), see [4-1] (Appendix B-6) and/or [4-2] (Appendix B-7). In fact, we show that such a surface has constant mean curvature: As a preliminary, we notice that

$$r = r(\theta) = \frac{a}{1 + e\cos\theta} \qquad (a > 0,\ 0 < e < 1) \tag{4.4}$$

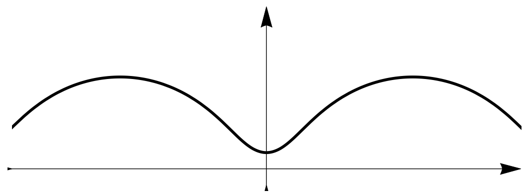

Figure 4: a nodoid

on the plane with respect to the polar coordinate system (r, θ) represents an ellipse such that O is one of the focal points, and e is its eccentricity. Since this ellipse can be parametrized as $\gamma(\theta) := r(\theta)(\cos\theta, \sin\theta)$, the tangent vector at $\mathrm{P} = \gamma(\theta)$ is computed as

$$\dot\gamma(\theta) \left(= \frac{d\gamma}{d\theta}\right) = \left(\frac{-a\sin\theta}{(1 + e\cos\theta)^2}, \frac{a(e + \cos\theta)}{(1 + e\cos\theta)^2}\right).$$

Let ξ be the angle between the vector $\overrightarrow{\mathrm{PO}}$ and the tangent of the ellipse at P (Fig. 5, left). Then we have

$$\cos\xi = \frac{-e\sin\theta}{\sqrt{1 + 2e\cos\theta + e^2}}, \qquad \sin\xi = \frac{1 + e\cos\theta}{\sqrt{1 + 2e\cos\theta + e^2}}.$$

We rotate the ellipse along the x-axis as in Fig. 5, right. When the ellipse has rotated angle θ about the focal point, then the point tangent to the x-axis has traveled the distance $s(\theta)$, which is the arc-length of $\gamma(\theta)$, that is,

$$s(\theta) = \int_0^\theta \frac{a\sqrt{1 + 2e\cos t + \varepsilon^2}}{(1 + e\cos t)^2}\, dt.$$

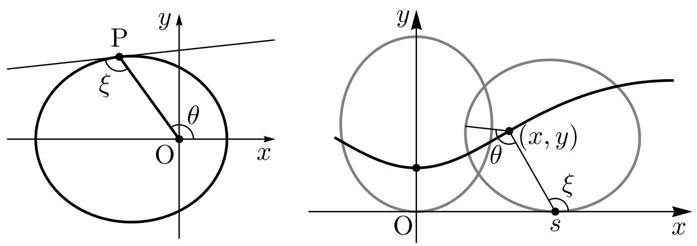

Figure 5:

Then the focal point of the ellipse is represented as

$$(x, y) = (x(\theta), y(\theta)) = (s(\theta) + r(\theta)\cos\xi(\theta), r(\theta)\sin\xi(\theta)).$$

Here, the mean curvature of the surface obtained by rotating the curve $(x(\theta), y(\theta))$ around the x-axis is computed as

$$H = \frac{\dot{y}\ddot{x} - \ddot{y}\dot{x}}{2(\sqrt{\dot{x}^2 + \dot{y}^2})^3} + \frac{\dot{x}}{2y\sqrt{\dot{x}^2 + \dot{y}^2}}.$$

Here,

$$\dot{x} = \frac{a(1 + e\cos\theta)}{\Delta^3}, \quad \dot{y} = \frac{ae\sin\theta}{\Delta^3}, \quad \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{a}{\Delta^2},$$

where $\Delta := \sqrt{1 + 2e\cos\theta + e^2}$. Hence we have $H = (1 - e^2)/(2a)$, which is a constant.

On the other hand, consider rotating a hyperbola along a line as in Fig. 6. Continuing the rotation, the tangent intersection of the hyperbola and the line moves out to infinity, and

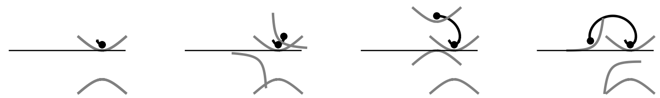

Figure 6:

the line tends to the asymptotic line of the hyperbola. From this limit state, we continue rotating the other component of the hyperbola along the given line. Repeating these over and over again, the locus of the focal point of the hyperbola is the generating curve of a nodoid. In fact, the polar equation of the ellipse (4.4), used in the case of an unduloid, also represents a hyperbola when $e > 1$. Then the mean curvature of the rotated surface can be computed similarly, which is constant.

References

[4-1] 梅原雅顕，山田光太郎，曲線と曲面（改訂版），裳華房，2014.

[4-2] Masaaki Umehara and Kotaro Yamada, Differential Geometry of Curves and Surfaces, (trasl. by Wayne Rossman), World Scientific, 2017.

Exercises

4-1 Explain what is a surfaces of revolution obtained by the locus of a focal point of an parabola while rolling it without slippage along a given line?