The Laplacian

Riemannian 2-manifolds. Let Σ be a 2 dimensional manifold. A *Riemannian metric* ds^2 of Σ is a collections of (positive definite) inner product of the tangent space $T_p\Sigma$ of Σ at p, here p runs over whole Σ. Then, for each $p \in \Sigma$, $(ds^2)_p$ is an inner product of the vector space $T_p\Sigma$. Let $(U; u, v)$ be a local coordinate system of Σ, then $\{\partial/\partial u, \partial/\partial v\}$ is a field of bases on U, namely, $\{(\partial/\partial u)_p, (\partial/\partial v)_p\}$ is a basis of $T_p\Sigma$ for each $p \in U$. We write the matrix representation of ds^2 with respect to such a field of bases as

$$\widehat{I} := \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \qquad \text{where} \quad \begin{aligned} E &= \left(\tfrac{\partial}{\partial u}, \tfrac{\partial}{\partial u}\right), \\ F &= \left(\tfrac{\partial}{\partial u}, \tfrac{\partial}{\partial v}\right), \\ G &= \left(\tfrac{\partial}{\partial v}, \tfrac{\partial}{\partial v}\right). \end{aligned} \tag{5.1}$$

Here, $(\ ,\ )$ denotes the inner product induced by ds^2. The Riemannian metric ds^2 is said to be *smooth* if E, F and G in (5.1) are smooth functions in (u, v). Note that this condition is independent of a choice of coordinate system. Throughout this section, Riemannian metrics are assumed to be smooth. Under the situation as in (5.1), we write

$$ds^2 := E\,du^2 + 2F\,du\,dv + G\,dv^2. \tag{5.2}$$

Lemma 5.1. *Let ds^2 in (5.2) be a Riemannian metric. Then*

$$E > 0, \qquad G > 0, \qquad \text{and} \qquad EG - F^2 > 0$$

holds.

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Proof. Since ds^2 is positive definite,

$$(\boldsymbol{v}, \boldsymbol{v}) = Ea^2 + 2Fab + Gb^2 > 0$$

holds for an arbitrary

$$\boldsymbol{v} := a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}.$$

In particular, letting $(a, b) = (1, 0)$ and $(0, 1)$, we have E, $G > 0$. Moreover, when $(a, b) = (-F, E)$, it holds that

$$0 < EF^2 - 2F^2E + E^2G = E(EG - F^2).$$

Then we have the conclusion. □

Assume the manifold Σ is oriented, and take a coordinate system $(U; u, v)$ on Σ which is compatible of the orientation. We call the differential 2-form

$$dA := \sqrt{EG - F^2}\,du \wedge dv \tag{5.3}$$

the *area element*.

Lemma 5.2. *The area element* (5.3) *does not depend on a choice of coordinate system compatible to the orientation.*

Proof. Let $(V; \xi, \eta)$ be another coordinate system such that the intersection with $(U; u, v)$ is not empty. Then

$$\left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}\right) = \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) J, \qquad J := \begin{pmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} & \frac{\partial v}{\partial \eta} \end{pmatrix}, \tag{5.4}$$

here we call J the *Jacobian matrix* of the coordinate change $(\xi, \eta) \mapsto (u, v)$. If we write

$$ds^2 = \widetilde{E}\, d\xi^2 + 2\widetilde{F}\, d\xi\eta^2 + \widetilde{G}\, dv^2,$$

E, F, G as in (**??**) and $\widetilde{E}$, $\widetilde{F}$, $\widetilde{G}$ are related as

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = {}^tJ \begin{pmatrix} \widetilde{E} & \widetilde{F} \\ \widetilde{F} & \widetilde{G} \end{pmatrix} J. \tag{5.5}$$

On the other hand,

$$\begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = J \begin{pmatrix} du \\ dv \end{pmatrix}. \tag{5.6}$$

Noticing $\det J > 0$ because (u, v) and (ξ, η) are compatible to the orientation, the conclusion follows by these equalities. □

Example 5.3. Let Σ is an oriented 2-manifold and $f\colon \Sigma \to \mathbb{R}^3$ an immersion. Then, for each $p \in \Sigma$, the restriction canonical inner product "$\cdot$" of $\mathbb{R}^3$ to $df(T_p\Sigma) \subset \mathbb{R}^3$ gives an inner product of $T_p\Sigma$, by identifying $T_p\Sigma$ and $df(T_p\Sigma)$. Thus, we have *the Riemannian metric ds^2 induced by the immersion f* which is nothing but the first fundamental form as in (1.2).

L^2-inner product for smooth functions. product Let (Σ, ds^2) be a Riemannian manifold, and assume that the manifold is oriented, for the sake of simplicity. We denote

$$\begin{aligned} C^\infty(\Sigma) &:= \text{the set of smooth functions on } \Sigma, \\ C_0^\infty(\Sigma) &:= \{\varphi \in C^\infty(\Sigma)\,;\, \operatorname{supp}\varphi \subset \Sigma \text{ is compact}\}, \end{aligned} \tag{5.7}$$

where $\operatorname{supp} f = \overline{\{p \in \Sigma\,;\, f(p) \neq 0\}}$ is the *support* of f. Then $C_0^\infty(\Sigma)$ is a linear subspace of the vector space $C^\infty(\Sigma)$.

Definition 5.4. The L^2-inner product of $C_0^\infty(\Sigma)$ is defined as

$$\langle \varphi, \psi \rangle := \int_\Sigma \varphi\psi\, dA \qquad (\varphi, \psi \in C_0^\infty(\Sigma))$$

where dA is the area element as in (5.3).

Then $\langle\ ,\ \rangle$ is an inner product of $C_0^\infty(\Sigma)$.

L^2-inner product of one forms. We denote

$$\varLambda^1(\Sigma) := \text{the set of smooth 1-forms on } \Sigma.$$

On a local coordinate system $(U; u, v)$, α, $\beta \in \varLambda^1(\Sigma)$ are expressed as

$$\alpha = \alpha_1\, du + \alpha_2\, dv, \qquad \beta = \beta_1\, du + \beta_2\, dv.$$

Then by (5.5) and (5.6),

$$(\alpha, \beta) := (\alpha_1, \alpha_2) \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \tag{5.8}$$

does not depend on a choice of coordinate system.

Definition 5.5. We denote

$$\varLambda_0^1(\Sigma) := \{\alpha \in \varLambda^1(\Sigma)\,;\, \operatorname{supp}\alpha \subset \Sigma \text{ is compact}\},$$

and define the *L^2-inner product* of one forms as

$$\langle \alpha, \beta \rangle := \int_\Sigma (\alpha, \beta)\, dA \qquad (\alpha, \beta \in \varLambda_0^1(\Sigma)).$$

Definition 5.6. For $\alpha \in \Lambda^1(\Sigma)$, we define

$$\delta\alpha := -\frac{1}{\sqrt{g}}\left[\left(\frac{G\alpha_1 - F\alpha_2}{\sqrt{g}}\right)_u + \left(\frac{-F\alpha_1 + E\alpha_2}{\sqrt{g}}\right)_v\right], \tag{5.9}$$

where $\alpha = \alpha_1\, du + \alpha_2\, dv$, E, F and G are as in (5.1), and $g := EG - F^2$.

Lemma 5.7. *The right-hand side of* (5.9) *does not depend on a choice of coordinate system.*

Proposition 5.8. *For* $\varphi \in C^\infty(\Sigma)$ *and* $\alpha \in \Lambda^1_0(\Sigma)$, *it holds that*

$$\langle \varphi, \delta\alpha\rangle = \langle d\varphi, \alpha\rangle \tag{5.10}$$

Proof. It is sufficient to show the equality when $\operatorname{supp} f$ and $\operatorname{supp}\alpha$ are contained in a local coordinate system $(U; u, v)$. In this case,

$$\begin{aligned}(d\varphi,\alpha)\, dA &= (d\varphi,\alpha)\sqrt{g}\, du\wedge dv\\ &= \frac{1}{\sqrt{g}}\left(\varphi_u(G\alpha_1 - F\alpha_2) + \varphi_v(-F\alpha_1 + E\alpha_2)\right)\\ &= d\omega + \delta\alpha\, dA\end{aligned}$$

hold for some one form ω, proving the conclusion. □

The Laplacian

Definition 5.9. The map $\Delta_{ds^2}\colon C^\infty(\Sigma) \to C^\infty(\Sigma)$ defined by

$$\Delta_{ds^2}\varphi := -\delta d\varphi \qquad (\varphi \in C^\infty(\Sigma))$$

is called the *Laplacian* with respect to the Riemannian metric ds^2.

Proposition 5.10. *For each* φ, $\psi \in \Lambda^1_0(\Sigma)$,

$$\int_\Sigma \varphi\Delta_{ds^2}\psi\, dA = -\langle d\varphi, d\psi\rangle$$

holds.

Proof. By Proposition 5.8,

$$\begin{aligned}\int_\Sigma \varphi\Delta_{ds^2}\psi\, dA &= \langle\varphi, \Delta_{ds^2}\psi\rangle = -\langle\varphi, \delta d\psi\rangle = -\langle d\varphi, d\psi\rangle\\ &= -\langle d\varphi, d\psi\rangle\end{aligned}$$

holds. □

A function $\varphi \in C^\infty(\Sigma)$ satisfying $\Delta_{ds^2}\varphi = 0$ is called a *harmonic function.*

Corollary 5.11. *A harmonic function on a compact, connected Riemannian manifold is a constant.*

Proof. Since Σ is compact, $C^\infty_0(\Sigma) = C^\infty(\Sigma)$. If φ is harmonic,

$$0 = \int_\Sigma \varphi\Delta_{ds^2}\varphi = -\langle d\varphi, d\varphi\rangle,$$

and hence $d\varphi = 0$. □

References

[5-1] 梅原雅顕・山田光太郎：曲線と曲面—微分幾何的アプローチ（改訂版），裳華房，2014.

Exercises

5-1[H] Consider the situation in Example 5.3, that is, $f\colon \Sigma \to \mathbb{R}^3$ be an immersion with the first fundamental form ds^2.

We write $f = (f_1, f_2, f_3)$, where f_j's ($j = 1, 2, 3$) are smooth functions defined on Σ. Then

$$\varDelta_{ds^2} f := (\varDelta_{ds^2} f_1, \varDelta_{ds^2} f_2, \varDelta_{ds^2} f_3)$$

is a vector valued function defined on Σ.

(1) Let $(U; u, v)$ be a local coordinate system of Σ. Show that $\varDelta_{ds^2} f$ is perpendicular to both f_u and f_v.

(2) Show that

$$\varDelta_{ds^2} f = 2H\nu,$$

where H and ν are the mean curvature and the unit normal vector field, respectively.

(3) An immersion f is said to be *minimal* if the mean curvature vanishes identically (see Definition 1.2). Prove that there are no compact minimal surface without boundary.

5-2[H] Let (Σ, ds^2) be a Riemannian 2-manifold. A coordinate system $(U; u, v)$ is said to be *isothermal* or *conformal* if the metric ds^2 is written as

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \tag{5.11}$$

where σ is a smooth function defined on U.

(1) Compute $\varDelta_{ds^2}\varphi$ with respect to the coordinate system (u, v).

(2) Let $(V; \xi, \eta)$ and $(U; u, v)$ are isothermal coordinate systems. Then the coordinate change

$$(\xi, \eta) \longmapsto (u(\xi, \eta), v(\xi, \eta))$$

satisfy

$$\varDelta_{ds^2} u = \varDelta_{ds^2} v = 0,$$

that is, coordinate changes between isothermal coordinate systems are harmonic.