Area minimizing surfaces

A review of surface theory.

Let $D \subset \mathbb{R}^2$ be a domain in the uv-plane and $f\colon D \to \mathbb{R}^3$ an immersion. We often refer to such an immersion as a *surface*. Then the *unit normal vector* of f is given by (with $\pm$-ambiguity)

$$\nu := \frac{f_u \times f_v}{|f_u \times f_v|} \colon D \longrightarrow S^2 = \{\boldsymbol{x} \in \mathbb{R}^3 \,|\, |\boldsymbol{x}| = 1\} \subset \mathbb{R}^3, \tag{1.1}$$

where "$\times$" denotes the vector product of $\mathbb{R}^3$. The *first* and the *second fundamental forms* are defined as

$$\begin{aligned} ds^2 &= df \cdot df = E\,du^2 + 2F\,du\,dv + G\,dv^2,\\ II &= -df \cdot d\nu = L\,du^2 + 2M\,du\,dv + N\,dv^2, \end{aligned} \tag{1.2}$$

where "$\cdot$" denotes the canonical inner product of $\mathbb{R}^3$. Here,

$$\begin{aligned} &E := f_u \cdot f_u, && F := f_u \cdot f_v = f_v \cdot f_u, && G := f_v \cdot f_v,\\ &L := -f_u \cdot \nu_u, && M := -f_u \cdot \nu_v = -f_v \cdot \nu_u, && N := -f_v \cdot \nu_v\\ &\quad = f_{uu} \cdot \nu, && \quad = f_{uv} \cdot \nu, && \quad = f_{vv} \cdot \nu \end{aligned}$$

are called the *entries of the first and the second fundamental forms* with respect to the parameters (u, v). The *area* of the image of a compact region $\Omega \subset D$ is computed as

$$\mathcal{A}(\Omega) := \iint_\Omega dA = \iint_\Omega |f_u \times f_v|\,du\,dv, \tag{1.3}$$

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where $dA = |f_u \times f_v|\,du\,dv = \sqrt{EG - F^2}\,du\,dv$ is said to be the *area element* of the surface.

The derivatives of ν is written as (the Weingarten Formula)

$$\nu_u = -A_1^1 f_u - A_1^2 f_v, \qquad \nu_v = -A_2^1 f_u - A_2^2 f_v, \tag{1.4}$$
$$A := \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

The matrix A is called the *Weingarten matrix*, and the determinant K and the half H of the trace of A are called the *Gaussian curvature* and the *mean curvature*, respectively:

$$K := \det A = \frac{LN - M^2}{EG - F^2}, \qquad H := \frac{1}{2} \operatorname{tr} A = \frac{A_1^1 + A_2^2}{2}. \tag{1.5}$$

Area minimizing surfaces.

The purpose of this section is to show the following fact:

> For a given simple closed curve C in $\mathbb{R}^3$, the surface which minimizing area among all surfaces bounded by C is a surface whose mean curvature vanishes identically.

Setting up. As the description of the above fact is rather intuituive, we will formulate the problem.

Let C be a simple closed smooth curve in $\mathbb{R}^3$ and set

$$\mathcal{S}_C := \left\{ f\colon \overline{D} \to \mathbb{R}^3\,;\ \begin{array}{l} f \text{ is a } C^\infty\text{-immersion} \\ f(\partial D) = C \end{array} \right\}, \tag{1.6}$$

where D (resp. $\overline{D}$) is the open (resp. closed) unit disc and ∂D is its boundary:[1]

$$(1.7)\quad \overline{D} := D \cup \partial D, \qquad \begin{aligned} D :&= \{(u,v) \in \mathbb{R}^2 \,;\, u^2+v^2<1\}, \\ \partial D :&= \{(u,v) \in \mathbb{R}^2 \,;\, u^2+v^2=1\} \\ &= \{(\cos\theta, \sin\theta)\,;\, \theta \in \mathbb{R}\}. \end{aligned}$$

Roughly speaking, $\mathcal{S}_C$ is "the set of the surfaces bounded by C". Then we set the *area functional* as

$$(1.8)\qquad \mathcal{A}\colon \mathcal{S}_C \ni f \longmapsto \mathcal{A}(f) = \iint_{\overline{D}} |f_u \times f_v| \, du\, dv.$$

Using these notations, our result can be stated as the following:

Theorem 1.1. *If a surface $f \in \mathcal{S}_C$ attains the minimum of the area functional $\mathcal{A}$, the mean curvature of f vanishes identically.*

Taking this fact into account, we define

Definition 1.2. A surface whose mean curvature vanishes identically is said to be *minimal*.

Remark 1.3. As Theorem 1.1 is a necessary condition for the minimizer, a minimal surface is not necessarily a minimizer of the area functional.

[1] A map f defined on $\overline{D}$ is said to be C^∞ if there exists a open set $\widetilde{D}$ containing $\overline{D}$ and a C^∞ map $\tilde{f}$ defined on $\widetilde{D}$ such that $\tilde{f}|_{\overline{D}} = f$.

Variations of surfaces. To show Theorem 1.1, we want to "differentiate" the functional $\mathcal{A}$.

Definition 1.4. For a surface $f \in \mathcal{S}_C$, a *variation* (fixing the boundary) of f is a C^∞-map

$$\mathcal{F}\colon \overline{D} \times (-\varepsilon, \varepsilon) \ni (u,v;t) \longmapsto f^t(u,v) := \mathcal{F}(u,v;t) \in \mathbb{R}^3$$

such that $f^0 = f$ and $f^t \in \mathcal{S}_C$ for each $t \in (-\varepsilon, \varepsilon)$, where ε is a positive number. The vector-valued function

$$(1.9)\qquad V(u,v) := \left.\frac{\partial}{\partial t}\right|_{t=0} f^t(u,v)$$

is called the *variational vector field* of the variation $\mathcal{F}$.

Lemma 1.5. *For a variation $\mathcal{F} = \{f^t\}$ of $f \in \mathcal{S}_c$ with variational vector field V, it holds that*

$$\frac{d}{d\theta} f(\cos\theta, \sin\theta) \times V(\cos\theta, \sin\theta) = \mathbf{0}.$$

Proof. Since $(\cos\theta, \sin\theta)$ is a parametrization of ∂D, $\gamma^t(\theta) := f^t(\cos\theta, \sin\theta) \in C$ for all t and θ. Thus, two vectors in the left-hand side of the first assertion are both tangent to C, proving the lemma. □

The first variation formula.

Theorem 1.6. *Let $\mathcal{F} = \{f^t\}$ be a variation of $f \in \mathcal{S}_C$ with variational vector field V. Then it holds that*

$$(1.10)\qquad \left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(f^t) = -2 \iint_{\overline{D}} H\bigl(V \cdot \nu\bigr)\, dA,$$

where H, ν and dA are the mean curvature, the unit normal vector and the area element of f, respectively.

Proof. By the definition of the area (5.3), we have

$$\begin{aligned}(*) := \left.\frac{d}{dt}\right|_{t=0}\mathcal{A}(f^t) &= \left.\frac{d}{dt}\right|_{t=0}\iint_{\overline{D}}|f_u^t\times f_v^t|\,du\,dv\\ &= \iint_{\overline{D}}\left.\frac{\partial}{\partial t}\right|_{t=0}|f_u^t\times f_v^t|\,du\,dv\\ &= \iint_{\overline{D}}\frac{\bigl(V_u\times f_v+f_u\times V_v\bigr)\cdot(f_u\times f_v)}{|f_u\times f_v|}\,du\,dv\\ &= \iint_{\overline{D}}\bigl(V_u\times f_v+f_u\times V_v\bigr)\cdot\nu\,du\,dv\\ &= \iint_{\overline{D}}\bigl(\bigl(V_u\times f_v\bigr)\cdot\nu+\bigl(f_u\times V_v\bigr)\cdot\nu\bigr)\,du\,dv.\end{aligned}$$

Here, by the formula of *scalar triple product*

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}=(\boldsymbol{b}\times\boldsymbol{c})\cdot\boldsymbol{a}=(\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{b}=\det(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}),$$

we have

$$\begin{aligned}(*) &= \iint_{\overline{D}}\bigl(\bigl(\nu\times f_v\bigr)\cdot V_u+\bigl(f_u\times\nu\bigr)\cdot V_v\bigr)\,du\,dv\\ &= (\mathrm{II})-(\mathrm{I}),\\ (\mathrm{I}) &:= \iint_{\overline{D}}\Bigl[\bigl(\bigl(\nu\times f_v\bigr)\cdot V\bigr)_u+\bigl(\bigl(f_u\times\nu\bigr)\cdot V\bigr)_v\Bigr]\,du\,dv,\\ (\mathrm{II}) &:= \iint_{\overline{D}}\Bigl[\bigl(\bigl(\nu\times f_v\bigr)_u\cdot V\bigr)+\bigl(f_u\times\nu\bigr)_v\cdot V\bigr)\bigr)\Bigr]\,du\,dv.\end{aligned}$$

By the Green-Stokes formula, (I) is computed as

$$\begin{aligned}(\mathrm{I}) &= \iint_{\overline{D}}\Bigl[\bigl(\bigl(\nu\times f_v\bigr)\cdot V\bigr)_u-\bigl(\bigl(\nu\times f_u\bigr)\cdot V\bigr)_v\Bigr]\,du\,dv,\\ &= \int_{\partial D}\nu\cdot\bigl(\bigl(f_u\,du+f_v\,dv\bigr)\times V\bigr)\\ &= \int_{-\pi}^{\pi}\nu\cdot\left(\frac{d}{d\theta}f(\cos\theta,\sin\theta)\times V(\cos\theta,\sin\theta)\right)d\theta=0.\end{aligned}$$

Here, the last assertion is obtained by Lemma 1.5. On the other hand, using the Weingarten formula (1.4), (II) is computed as

$$\begin{aligned}(\mathrm{II}) := &\iint_{\overline{D}}\Bigl[\bigl(\nu_u\times f_v\bigr)\cdot V+\bigl(\nu\times f_{vu}\bigr)\cdot V\\ &\qquad+\bigl(f_{uv}\times\nu\bigr)\cdot V+\bigl(f_u\times\nu_v\bigr)\cdot V\Bigr]\,du\,dv\\ =&\iint_{\overline{D}}\Bigl[\bigl(\nu_u\times f_v\bigr)\cdot V+\bigl(f_u\times\nu_v\bigr)\cdot V\Bigr]\,du\,dv\\ =&-\iint_{\overline{D}}\Bigl[\bigl((A_1^1f_u+A_1^2f_v)\times f_v\bigr)\cdot V\\ &\qquad+\bigl(f_u\times(A_2^1f_u+A_2^2f_v)\bigr)\cdot V\Bigr]\,du\,dv\\ =&-\iint_{\overline{D}}(A_1^1+A_2^2)(f_u\times f_v)\cdot V\,du\,dv\\ =&-\iint_{\overline{D}}2H(\nu\cdot V)|f_u\times f_v|\,du\,dv\end{aligned}$$
□

Proof of Theorem 1.1. We need the following "the fundamental lemma for calculus of variations".

Lemma 1.7. *Assume a smooth function* $h\colon \overline{D} \to \mathbb{R}$ *satisfies*

$$\iint_{\overline{D}} h(u,v)\varphi(u,v)\,du\,dv = 0$$

for all C^∞*-function with* $\varphi|_{\partial D} = 0$*. Then* $h = 0$ *on* D*.*

Proof. Assume $h(u_0, v_0) > 0$ (resp. < 0)($(u_0, v_0) \in D$). By a continuity, there exists $\varepsilon > 0$ such that $h(u,v) > -$ on an ε-ball $B := B_\varepsilon(u_0, v_0)$ centered at (u_0, v_0). Let φ be a non-negative C^∞-function on $\overline{D}$ such that $\varphi > 0$ on B and 0 on $\overline{D} \setminus B$. Then

$$\iint_{\overline{D}} h\varphi\,du\,dv = \iint_B h\,\varphi\,du\,dv > 0 \qquad (\text{resp.} < 0),$$

a contradiction. □

Proof of Theorem 1.6. Assume $f \in \mathcal{S}_C$ minimizes the area. Then for any variation $\mathcal{F} = \{f^t\}$ of f, $\mathcal{A}(f^t)$ is not less than $\mathcal{A}(f) = \mathcal{A}(f^0)$. Then by Theorem 1.6, it holds that

$$0 = \left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(f^t) = -2\int_{\overline{D}} H(V\cdot\nu)|f_u \times f_v|\,du\,dv.$$

Let φ be a C^∞-function on $\overline{D}$ with $\varphi|_{\partial D} = 0$. Then $f^t := f + t\varphi\nu$ is a variation of f with variational vector field $V = \varphi\nu$. Thus,

$$\iint H|f_u \times f_v|\varphi = 0.$$

Since φ is arbitrary, Lemma 1.7 yields the conclusion. □

Exercises

1-1[H] Define a functional $\mathcal{V}\colon \mathcal{S}_C \to \mathbb{R}$ defined on $\mathcal{S}_C$ as in (1.6) as

$$\mathcal{V}(f) := \frac{1}{3}\iint_{\overline{D}} (f\cdot\nu)|f_u \times f_v|\,du\,dv \qquad (f \in \mathcal{S}_C).$$

Then

(1) Explain a geometric meaning of $\mathcal{V}(f)$.

(2) Compute $\left.\frac{d}{dt}\right|_{t=0} \mathcal{V}(f^t)$ for a variation $\{f^t\}$ of $f \in \mathcal{S}_C$.

Surfaces of constant mean curvature

Closed surfaces A *closed surface* in the Euclidean 3-space $\mathbb{R}^3$ is a C^∞-immersion $f\colon \Sigma \to \mathbb{R}^3$ of a compact, connected 2 dimensional manifold Σ into $\mathbb{R}^3$. Taking a local coordinate neighborhood $(U; u, v)$ of Σ, f can be identified a parametrized surface $f(u, v)$ as in the previous section.

Throughout this section, we assume that Σ is *oriented*, that is, an atlas $\{(U_\alpha; u^\alpha, v^\alpha) \,|\, \alpha \in A\}$ of Σ satisfying

$$\frac{\partial(u^\beta, v^\beta)}{\partial(u^\alpha, v^\alpha)} := \det J_{\alpha\beta} > 0 \qquad \text{on } U_\alpha \cap U_\beta \tag{2.1}$$

for each α, $\beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$ is specified. Here $J_{\alpha\beta}$ is the *Jacobian matrix* of the coordinate change $(u^\alpha, v^\alpha) \mapsto (u^\beta, v^\beta)$

$$J_{\alpha\beta} := \begin{pmatrix} \frac{\partial u^\beta}{\partial u^\alpha} & \frac{\partial u^\beta}{\partial v^\alpha} \\ \frac{\partial v^\beta}{\partial u^\alpha} & \frac{\partial v^\beta}{\partial v^\alpha} \end{pmatrix} \tag{2.2}$$

Fix a coordinate neighborhood $(U; u, v)$. Then the immersion $f\colon (u, v) \mapsto f(u, v)$ is considered as a vector-valued smooth function on U, and so are there derivatives f_u and f_v. Then the unit normal vector ν, the first fundamental form ds^2, the second fundamental form $I\!I$, the area element dA, the Gaussian curvature K and the mean curvature H are defined as in (1.1), (1.2), (5.3) and (1.5) in the previous section. Moreover, one can prove easily that they are independent on choice of local coordinate systems (cf. [2-1] and/or [2-2]).

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Definition 2.1. Let $f\colon \Sigma \to \mathbb{R}^3$ be an oriented closed surface. Then the *area* $\mathcal{A}(f)$ of $f(\Sigma)$ and the (signed) *volume* $\mathcal{V}(f)$ of the region bounded by $f(\Sigma)$ are defined as

$$\mathcal{A}(f) := \int_\Sigma dA, \qquad \mathcal{V}(f) := \frac{1}{3}\int_\Sigma f \cdot \nu \, dA,$$

where "$\cdot$" denotes the canonical inner product of $\mathbb{R}^3$, ν is the unit normal vector as in (1.1), and dA denotes the area element which is represented by $dA := |f_u \times f_v| \, du\, dv$ on each coordinate neighborhood $(U; u, v)$.

Remark 2.2. If the surface f is an embedding, that is, the map f is injective (in this case), the image $f(\Sigma)$ bounds a bounded and connected region D of $\mathbb{R}^3$, and the volume of D coincide with the absolute value of $\mathcal{V}(f)$.

Obviously, these two functionals have the following properties:

Lemma 2.3. *For an immersion $f \in \mathcal{S}(\Sigma)$ and a positive number $\lambda > 0$, $\mathcal{A}(\lambda f) = \lambda^2 \mathcal{A}(f)$, and $\mathcal{V}(\lambda f) = \lambda^3 \mathcal{V}(f)$ hold.*

Example 2.4 (The round sphere)**.** Let $R > 0$ be a constant and denote by

$$S^2(R) := \left\{ \boldsymbol{x} \in \mathbb{R}^3 \,|\, |\boldsymbol{x}| = R \right\} \subset \mathbb{R}^3$$

the sphere in $\mathbb{R}^3$ of radius R centered at the origin. Then the inclusion map

$$\iota\colon S^2(R) \ni \boldsymbol{x} \longmapsto \iota(\boldsymbol{x}) = \boldsymbol{x} \in \mathbb{R}^3$$

is an embedding. A map

$$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\times(-\pi,\pi)\ni(u,v) \\ \longmapsto (R\cos u\cos v, R\cos u\sin v, R\sin u)\in S^2(R)$$

gives a local coordinate system of $S^2(R)$, and we have

$$dA = R^2\cos u\,du\,dv, \qquad \nu = -(\cos u\cos v, \cos u\sin v, \sin u).$$

Since this coordinate neighborhood covers an open dense subset of $S^2(R)$, "integration over $S^2(R)$" is replaced by "integration over $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[-\pi,\pi]$":

$$\begin{aligned}\mathcal{A}(\iota) &= \int_{-\pi/2}^{\pi/2} du \int_{-\pi}^{\pi} dv\, R^2\cos u \\ &= 2\pi R^2\int_{-\pi/2}^{\pi/2}\cos u\,du = 4\pi R^2, \\ \mathcal{V}(\iota) &= \frac{1}{3}\int_{-\pi/2}^{\pi/2}\int_{-\pi}^{\pi} R^3\cos u\,du\,dv = -\frac{4}{3}\pi R^3.\end{aligned}$$

The Gaussian and the mean curvature are computed as

$$K = \frac{1}{R^2} \qquad \text{and} \qquad H = \frac{1}{R},$$

respectively, which are constant on the surface. We call S^2_R the *round sphere* of radius R.

Area minimizing surfaces with a volume constraint. Let Σ be a compact, connected and oriented 2-manifold and consider

$$\mathcal{S}(\Sigma) = \{f\colon \Sigma\to\mathbb{R}^3 \mid f \text{ is an immersion}\}. \tag{2.3}$$

In addition, for a fixed positive constant V_0. we set

$$\mathcal{S}(\Sigma, V_0) := \{f\in\mathcal{S}(\Sigma) \mid \mathcal{V}(f) = V_0\}, \tag{2.4}$$

that is, $\mathcal{S}(\Sigma, V_0)$ is the set of immersions of Σ into $\mathbb{R}^3$ bounding given volume V_0.

In this section, we shall prove

Theorem 2.5. *If $f_0\in\mathcal{S}(\Sigma,V_0)$ minimizes the area in $\mathcal{S}(\Sigma,V_0)$, the mean curvature of f_0 is non-zero constant.*

Theorem 2.5 and Example 2.4 give rise to the following question, known as Heinz-Hopf's problem:

Question 2.6. *Are there a closed surface of constant mean curvature which is not congruent to the round sphere?*

Variation formula for the area and the volume Similar to the previous section, we define variations of $f\in\mathcal{S}(\Sigma)$:

Definition 2.7. A *variation* of an immersion $f\colon\Sigma\to\mathbb{R}^3$ is a C^∞-map $F\colon(-\varepsilon,\varepsilon)\times\Sigma\to\mathbb{R}^3$ satisfying

- $f^t := F(t,*)\colon\Sigma\to\mathbb{R}^3$ is an immersion for each $t\in(-\varepsilon,\varepsilon)$,
- $f^0 = F(0,*)$ coincides with f.

The variational vector field V of a variation $F = \{f^t\}$ is a vector-valued function V on Σ defined by

$$V(p) := \left.\frac{\partial}{\partial t}\right|_{t=0} F(t,p) \qquad (p \in \Sigma).$$

Similar to variational formula in Section 1, we have

Theorem 2.8. *Let $\{f^t\}$ be a variation of an immersion $f\colon \Sigma \to \mathbb{R}^3$. Then*

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(f^t) = -2\int_\Sigma H\varphi\, dA, \qquad \left.\frac{d}{dt}\right|_{t=0} \mathcal{V}(f^t) = \int_\Sigma \varphi\, dA,$$

hold, where $\varphi := V\cdot\nu$, V is the variational vector field of $\{f^t\}$ and ν is the unit normal vector field of f.

Proof. Since almost all part of the computation in the previous section are coordinate-independent, we can show the result in a similar way to them.

Here, we shall prove the formula for the volume functional. Let $(U;u,v)$ be a local coordinate system. Then it holds that

$$\begin{aligned}\varPhi := f^t\cdot\nu^t|f^t_u\times f^t_v| &= f^t\cdot\frac{f^t_u\times f^t_v}{|f^t_u\times f^t_v|}\,|f^t_u\times f^t_v| \\ &= \det(f^t, f^t_u, f^t_v)\end{aligned}$$

Differentiating this in t, we have

$$\begin{aligned}\left.\frac{\partial}{\partial t}\right|_{t=0}\varPhi &= \det(\dot f^t, f_u, f_v) + \det(f, \dot f^t_u, f_v) + \det(f, f_u, \dot f^t_v) \\ &= \det(V, f_u, f_v) + \det(f, V_u, f_v) + \det(f, f_u, V_v),\end{aligned}$$

where $\dot{*} = (\partial/\partial t)|_{t=0}$. Here, since

$$\begin{aligned}\det(V, f_u, f_v) &= V\cdot(f_u\times f_v) = (V\cdot\nu)|f_u\times f_v|,\\ \det(f, V_u, f_v) &= \big(\det(f,V,f_v)\big)_u - \det(f,V,f_{uv}) - \det(f_u,V,f_v)\\ &= \big(\det(f,V,f_v)\big)_u - \det(f,V,f_{uv}) + \det(V,f_u,f_v)\\ \det(f, f_u, V_v) &= \big(\det(f,f_u,V)\big)_v - \det(f,f_{uv},V) - \det(f_v,f_u,V)\\ &= \big(\det(f,f_u,V)\big)_v - \det(f,f_{uv},V) + \det(V,f_u,f_v),\end{aligned}$$

it holds that

$$\begin{aligned}\left(\left.\frac{\partial}{\partial t}\right|_{t=0}\varPhi\right) du\wedge dv &= 3(V\cdot\nu)\,|f_u\times f_v|\,du\wedge dv\\ &\quad + \Big(\big(\det(f,V,f_v)\big)_u + \big(\det(f,f_u,V)\big)_v\Big)\,du\wedge dv.\end{aligned}$$

Here, setting

$$\alpha := \det(f,V,f_u)\,du + \det(f,V,f_v)\,dv = \det(f,V,df),$$

we have the coordinate-independent expression

$$\left(\left.\frac{\partial}{\partial t}\right|_{t=0}\varPhi\right) du\wedge dv = 3(V\cdot\nu)\,dA + d\alpha,$$

and then,

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0}\mathcal{V}(f^t) &= \frac{1}{3}\int_\Sigma\left(\left.\frac{\partial}{\partial t}\right|_{t=0}\varPhi\right) du\wedge dv\\ &= \int_\Sigma (V\cdot\nu)\,dA + \frac{1}{3}d\alpha = \int_\Sigma (V\cdot\nu)\,dA,\end{aligned}$$

proving the formula. □

Proof of Theorem 2.5. Let $f_0 \in \mathcal{S}(\Sigma, V_0)$ be an immersion minimizing area in $\mathcal{S}(\Sigma, V_0)$. Then it holds that

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(f^t) = 0 \qquad \text{for any volume preserving variation } \{f^t\}. \tag{2.5}$$

Here, a variation $\{f^t\}$ of f_0 is said to be *volume preserving* if $\mathcal{V}(f^t) = \mathcal{V}(f_0)$ for all t.

Let $\{f^t\}$ be a (not necessarily volume preserving) variation of f_0. Then, by Lemma 2.3, $\{\tilde{f}^t\}$ defined by

$$\tilde{f}^t := \frac{\mathcal{V}(f^t)^{-1/3}}{\mathcal{V}(f_0)^{-1/3}}\, f^t$$

is volume preserving variation, and the map $\{f^t\} \mapsto \{\tilde{f}^t\}$ is a surjection to the set of volume preserving variations. That is, (2.5) is equivalent to

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{A}\left(\frac{\mathcal{V}(f^t)^{-1/3}}{\mathcal{V}(f_0)^{-1/3}}\, f^t\right) = 0 \quad \text{for any variation } \{f^t\}. \tag{2.6}$$

Here, by Theorem 2.8,

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \mathcal{A}\bigl(\mathcal{V}(f^t)^{-1/3} f^t\bigr) &= \left.\frac{d}{dt}\right|_{t=0} \mathcal{V}(f^t)^{-2/3}\mathcal{A}(f^t)\\
&= -\frac{2}{3}\dot{\mathcal{V}}(f^t)\mathcal{V}(f_0)^{-5/3}\mathcal{A}(f_0) + \mathcal{V}(f_0)^{-2/3}\dot{\mathcal{A}}(f^t)\\
&= \mathcal{V}(f_0)^{-2/3}\left(-\frac{2}{3}\frac{\mathcal{A}(f_0)}{\mathcal{V}(f_0)}\dot{\mathcal{V}}(f^t) + \dot{\mathcal{A}}(f^t)\right)\\
&= \mathcal{V}(f_0)^{-2/3}\left(\int_\Sigma \left(-\frac{2}{3}\frac{\mathcal{A}(f_0)}{\mathcal{V}(f_0)} - 2H\right)\varphi\, dA\right),
\end{aligned}$$

where $\dot{*} = (d/dt)|_{t=0}$ and $\varphi = V \cdot \nu$. Then by Lemma 1.7,

$$-\frac{2}{3}\frac{\mathcal{A}(f_0)}{\mathcal{V}(f_0)} - 2H = 0,$$

holds, and then H is constant.

References

[2-1] 梅原雅顕，山田光太郎，曲線と曲面（改訂版），裳華房，2014.

[2-2] Masaaki Umehara and Kotaro Yamada, Differential Geometry of Curves and Surfaces, (trasl. by Wayne Rossman), World Scientific, 2017.

Exercises

2-1[H] Let $\mathcal{C} := \{\gamma\colon S^1 \to \mathbb{R}^2 \,|\, \gamma' \neq \mathbf{0}\}$ be the set of regular closed curves on $\mathbb{R}^2$.

(1) Define the area $\mathcal{A}(\gamma)$ of the region bounded by γ.

(2) Let $\mathcal{C}(a)$ be the set of curves γ with $\mathcal{A}(\gamma) = a$. Show that if a curve $\gamma_0 \in \mathcal{C}(a)$ minimizes the length in $\mathcal{C}(a)$, the curvature of γ_0 is constant.

Hint: A curve $\gamma \in \mathcal{C}(a)$ can be parametrized $\gamma(t) = {}^t(x(t), y(t))$ as a 2π-periodic function. The length $\mathcal{L}(\gamma)$ and the curvature function κ of γ are defined as

$$\mathcal{L}(\gamma) := \int_0^{2\pi} |\dot{\gamma}(t)|\, dt, \qquad \kappa(t) := \frac{\det(\dot{\gamma}(t), \ddot{\gamma}(t))}{|\dot{\gamma}(t)|^3}$$

where $\dot{} = d/dt$.

Examples of Constant Mean Curvature Surfaces

Planar curves. Let $\gamma\colon I \ni s \mapsto \gamma(s) \in \mathbb{R}^2$ be a smooth map defined on an $I \subset \mathbb{R}$. Then γ is called a *regular curve* if $\dot\gamma \neq 0$ on I, where $\dot{} = d/ds$. The parameter s is called an *arc length parameter* if

$$|\dot\gamma(s)| = \left|\frac{d\gamma}{ds}(s)\right| = 1 \tag{3.1}$$

holds on I.

Lemma 3.1. *A regular curve $\gamma\colon I \ni t \mapsto \gamma(t) \in \mathbb{R}^2$ defined on an interval $I \subset \mathbb{R}$ can be reparametrized by an arc length parameter. Moreover, such an arc length parameter is unique up to additive constants.*

Proof. Fix $t_0 \in I$ and define a function $s\colon I \to \mathbb{R}$ by

$$s(t) := \int_{t_0}^{t} \left|\frac{d\gamma}{dt}(u)\right| du.$$

Then $s\colon I \to J \subset \mathbb{R}$ is a smooth function such that $ds/dt > 0$. Hence there exists the smooth inverse $J \ni s \mapsto t(s) \in I$. Then $\tilde\gamma(s) := \gamma\bigl(t(s)\bigr)$ is the desired reparametrization. In fact,

$$\left|\frac{d\tilde\gamma(s)}{ds}\right| = \left|\frac{d\gamma}{dt}\bigl(t(s)\bigr)\frac{dt}{ds}(s)\right| = \left|\frac{d\gamma}{dt}\bigl(t(s)\bigr)\frac{1}{ds/dt(t(s))}\right| = \left|\frac{d\gamma}{dt}\bigl(t(s)\bigr)\frac{1}{|d\gamma/dt\bigl(t(s)\bigr)|}\right| = 1.$$

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So we have the first assertion. Let s and u be two arc length parameters. Then there exists a parameter change $u = u(s)$, which is strictly increasing function such that

$$1 = \left|\frac{d\gamma}{ds}\right| = \left|\frac{d\gamma}{du}\frac{du}{ds}\right| = \frac{du}{ds}\left|\frac{d\gamma}{du}\right| = \frac{du}{ds}.$$

Hence $u = s + \text{constant}$, proves the second assertion. □

Throughout this section, we assume that planar curves are parameterized by arc length parameter.

Let $\gamma(s) = {}^t\bigl(x(s), y(s)\bigr)$ $(s \in I)$be a parametrized planar curve where s is an arc length parameter. Then

$$\boldsymbol{e}(s) := \dot\gamma(s) = \begin{pmatrix}\dot x(s)\\ \dot y(s)\end{pmatrix}, \qquad \boldsymbol{n}(s) := \begin{pmatrix}-\dot y(s)\\ \dot x(s)\end{pmatrix}$$

are mutually perpendicular orthogonal vectors for each $s \in I$. Thus we have obtained a map

$$\mathcal{F}(s) := \bigl(\boldsymbol{e}(s), \boldsymbol{n}(s)\bigr)\colon I \longmapsto \mathrm{SO}(2), \tag{3.2}$$

where $\mathrm{SO}(2)$ is the set (a group) of 2×2-orthogonal matrix of determinant 1. We call $\mathcal{F}$ the *frame* of γ. Note that

$$\mathrm{SO}(2) = \{R(\theta)\,|\,\theta \in \mathbb{R}\}, \quad R(\theta) := \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}. \tag{3.3}$$

Theorem 3.2 (The Frenet formula)**.** *Let $\mathcal{F}(s)$ be the frame of the curve $\gamma(s)$ where s is an arc length parameter defined on an* *interval I. Then there exists a unique smooth function $\kappa\colon I \to \mathbb{R}$ such that*

$$\dot{\mathcal{F}} = \mathcal{F}\Omega \qquad \Omega(s) := \kappa(s)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.4}$$

Proof. Since $\mathcal{F}$ is a function valued on SO(2), $\mathcal{F}^{-1}\dot{\mathcal{F}}$ is valued on the set of skew-symmetric matrices. In fact, since ${}^t\mathcal{F} = \mathcal{F}^{-1}$,

$$\begin{aligned} {}^t\left(\mathcal{F}^{-1}\dot{\mathcal{F}}\right) &= {}^t\left({}^t\mathcal{F}\dot{\mathcal{F}}\right) = {}^t\dot{\mathcal{F}}\mathcal{F} = \frac{d}{ds}\mathcal{F}^{-1}\mathcal{F} \\ &= -\mathcal{F}^{-1}\dot{\mathcal{F}}\mathcal{F}^{-1}\mathcal{F} = -\mathcal{F}^{-1}\dot{\mathcal{F}}. \end{aligned}$$

Hence there exists a function $\kappa(s)$ such that

$$\mathcal{F}^{-1}\dot{\mathcal{F}} = \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix},$$

proving the theorem. □

We call the function κ the *curvature* of the curve γ.

Proposition 3.3. *Let $\gamma(s) = {}^t\big(x(s), y(s)\big)$ be a planar curve parametrized by the arc length s. Then its curvature satisfies*

$$\kappa = \dot{x}\ddot{y} - \dot{y}\ddot{x}.$$

Theorem 3.4 (The fundamental theorem for planar curves)**.** *Let $\kappa\colon I \to \mathbb{R}$ be a smooth function. Then there exists a curve $\gamma\colon I \to \mathbb{R}$ parametrized by the arc length whose curvature is κ. Moreover, such a curve γ is unique up to rotations and translations of $\mathbb{R}^2$.*

Proof. First we shall prove uniqueness: Let γ_j $(j = 1, 2)$ be curves with curvature κ, and denote by $\mathcal{F}_j$ $(j = 1, 2)$ the frame of γ_j. Then by (3.4),

$$\begin{aligned} \frac{d}{ds}(\mathcal{F}_2\mathcal{F}_1^{-1}) = \frac{d}{ds}(\mathcal{F}_2{}^t\mathcal{F}_1) &= \dot{\mathcal{F}}_2{}^t\mathcal{F}_1 + \mathcal{F}_2{}^t\dot{\mathcal{F}}_1 \\ &= \mathcal{F}_2\Omega{}^t\mathcal{F}_1 + \mathcal{F}_2{}^t(\mathcal{F}_1\Omega) = \mathcal{F}_2(\Omega + {}^t\Omega){}^t\mathcal{F}_1 = O \end{aligned}$$

holds, and thus there exists a constant matrix such that

$$\mathcal{F}_2\mathcal{F}_1^{-1} = A \qquad (A \in \mathrm{SO}(2)),$$

that is, $\mathcal{F}_2 = A\mathcal{F}_1$. Comparing the first column of this, we have

$$\dot{\gamma}_2 = A\dot{\gamma}_1 \qquad \text{and then} \qquad \gamma_2 = A\gamma_1 + \boldsymbol{a},$$

where $A \in \mathrm{SO}(2)$ and $\boldsymbol{a} \in \mathbb{R}^2$. Hence the uniqueness part holds.

Next, we prove existence: fix $s_0 \in I$ and set

$$\gamma(s) := \int_{s_0}^{s} \left(\cos\int_{s_0}^{u}\kappa(t)\,dt, \sin\int_{s_0}^{u}\kappa(t)\,dt\right) du.$$

Then one can check that s is the arc length parameter of $\gamma(s)$, and $\kappa(s)$ is the curvature. □

Surfaces of revolution. Let $\gamma(s) = \big(x(s), y(s)\big)$ be a regular curve parametrized by the arc length s, satisfying $y(s) > 0$ for all s. Then the *surface of revolution* of γ about the x-axis is parametrized as

$$f(t, s) := \big(x(s), y(s)\cos t, y(s)\sin t\big), \qquad (t, s) \in S^1 \times I. \tag{3.5}$$

The curve γ is called the *profile curve* of the surface (3.5).

Noticing $\dot{x}^2 + \dot{y}^2 = 1$, the first fundamental form I and the second fundamental form II of f are expressed as

$$I = y^2\,dt^2 + ds^2, \quad II = -\dot{x}y\,dt^2 + (\dot{x}\ddot{y} - \dot{y}\ddot{x})\,ds^2 = -\dot{x}y\,dt^2 + \kappa\,ds^2,$$

where κ is the curvature of the profile curve (cf. Proposition 3.3). Hence we have

Proposition 3.5. *The mean curvature function H of the surface* (3.5) *is expressed as*

$$2H = \kappa - \frac{\dot{x}}{y}. \tag{3.6}$$

Delaunay surfaces.

Theorem 3.6. *Let H be a non-zero constant. Then the profile curve $(x(s), y(s))$ of a surface of revolution with constant mean curvature H is expressed as*

$$\begin{aligned} y(s) &= \frac{1}{2|H|}\sqrt{(2Ha+1)^2 - 2(2Ha+1)\cos 2Hs + 1}, \\ x(s) &= \int_0^s \frac{(2Ha+1)\cos 2Hu - 1}{2Hy(u)}\,du, \end{aligned} \tag{3.7}$$

up to horizontal translations and parameter changes, where a is a constant.

Proof. Let $\gamma(s) := \bigl(x(s), y(s)\bigr)$ be the profile curve of given surface of revolution with constant mean curvature H. Then by (3.6), the curvature function κ of γ satisfies

$$\kappa = 2H + \frac{\dot{x}}{y}.$$

Thus, the frame $\mathcal{F}$ of γ satisfies the Frenet formula (Theorem 3.2):

$$\dot{\mathcal{F}} = \left(2H + \frac{\dot{x}}{y}\right)\mathcal{F}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.8}$$

We shall find the curve solving this differential equation. Set

$$\widetilde{\mathcal{F}} := y\mathcal{F}.$$

Then, noticing

$$\dot{x}^2 + \dot{y}^2 = 1, \tag{3.9}$$

the equation (3.8) is equivalent to

$$\dot{\widetilde{\mathcal{F}}} = \widetilde{\mathcal{F}}\begin{pmatrix} 0 & -2H \\ 2H & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.10}$$

Let

$$\begin{aligned} &A(s) := \widetilde{\mathcal{F}}(s)\mathcal{F}_0(s)^{-1}, \\ &\mathcal{F}_0(s) := R(2Hs) = \begin{pmatrix} \cos 2Hs & -\sin 2Hs \\ \sin 2Hs & \cos 2Hs \end{pmatrix}. \end{aligned} \tag{3.11}$$

Substituting $\widetilde{\mathcal{F}} = A\mathcal{F}_0$ into (3.10), we have

$$\dot{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathcal{F}_0^{-1} = \begin{pmatrix} \sin 2Hs & -\cos 2Hs \\ \cos 2Hs & \sin 2Hs \end{pmatrix}, \tag{3.12}$$

and then

$$A = \frac{-1}{2H}\begin{pmatrix} \cos 2Hs & \sin 2Hs \\ -\sin 2Hs & \cos 2Hs \end{pmatrix} + C, \tag{3.13}$$

where C is a constant matrix. Summing up, it holds that

$$\begin{aligned} y\mathcal{F} = \widetilde{\mathcal{F}} &= A\mathcal{F}_0 \\ &= \frac{1}{2H}\left(C\begin{pmatrix} \cos 2Hs & -\sin 2Hs \\ \sin 2Hs & \cos 2Hs \end{pmatrix} - \mathrm{id}\right). \end{aligned} \tag{3.14}$$

Since right-hand side is a periodic function and $\mathcal{F} \in \mathrm{SO}(2)$, y^2 (and then y) is a periodic function. Hence y must take both maximum and minimum. By a change of parameter s to $s +$ constant and a horizontal translation $x \mapsto x +$ constant, we may assume y takes its maximum at $s = 0$, and $x(0) = 0$. Moreover, by the reflection of the y-axis, we may assume $\dot{x}(0) \geqq 0$ without loss of generality.[2] Hence we can assume an initial condition

$$\bigl(x(0), y(0)\bigr) = (0, a), \quad \bigl(\dot{x}(0), \dot{y}(0)\bigr) = (1, 0), \quad \ddot{y}(0) = \kappa(0) \leqq 0.$$

Substituting these into (3.14), we have $C = (2Ha + 1)\,\mathrm{id}$:

$$y\mathcal{F} = \frac{1}{2H}\left((2Ha+1)\begin{pmatrix} \cos 2Hs & -\sin 2Hs \\ \sin 2Hs & \cos 2Hs \end{pmatrix} - \mathrm{id}\right). \tag{3.15}$$

Taking the determinant of this, we have

$$y^2 = \frac{1}{(2H)^2}\bigl((2Ha+1)\cos 2Hs - 1\bigr)^2 + (2Ha+1)^2 \sin^2 2Hs\bigr)$$

[2]Note that H changes its sign by a reflectoin. Under these assumptions, H must be non-positive because of (3.6).

and then

$$y = \frac{1}{2|H|}\sqrt{(2Ha+1)^2 - 2(2Ha+1)\cos 2Hs + 1}.$$

On the other hand, the $(1,1)$-component of (3.15) is expressed as

$$y\dot{x} = \frac{1}{2H}\bigl((2aH+1)\cos 2Hs - 1\bigr).$$

Thus we have the conclusion when $H < 0$. By replacing s by $-s$, the mean curvature changes the sign. Hence the same expressions are obtained. □

The surfaces in (3.7) are called the *Delaunay surfaces*.

References

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[3-2] K. Kenmotsu, Surfaces with constant mean curvature, Translations of Mathematical Monographs, translated by Katsuhiro Moriya, American Math. Soc., 2003.

Exercises

3-1[H] Draw pictures of Delaunay curves for $H = \frac{1}{2}$.

3-2[H] Classify minimal surfaces of revolution.

Delaunay Surfaces

Constant mean curvature surfaces of revolution. As seen in Theorem 3.6 in the previous section, we have

Theorem 4.1. *Let H and a be arbitrary constants with*

$$H > 0 \qquad \text{and} \qquad 2Ha + 1 > 0, \tag{4.1}$$

and let $\gamma(s) = \bigl(x(s), y(s)\bigr)$ with

$$\begin{aligned} y(s) &= \frac{1}{2|H|}\sqrt{(2Ha+1)^2 - 2(2Ha+1)\cos 2Hs + 1},\\ x(s) &= \int_0^s \frac{(2Ha+1)\cos 2Hu - 1}{2Hy(u)}\,du. \end{aligned} \tag{4.2}$$

Then the surface of revolution with respect to the x axis with profile curve γ has constant mean curvature H. Conversely, constant mean curvature surfaces of revolution are obtained in this manner.

Proof. Take $H \neq 0$ and a arbitrarily. Then one can easily show that the surface of revolution with profile curve $(x(s), y(s))$ has constant mean curvature H.

On the other hand, in the proof of Theorem 3.6, we have solved the differential equation (3.8) with initial condition

$$\bigl(x(0), y(0)\bigr) = (0, a), \quad \bigl(\dot x(0), \dot y(0)\bigr) = (1, 0), \quad \ddot y(0) = \kappa(0) \leqq 0,$$

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where a is a positive constant. In this case, (3.6) yields that

$$2H = \kappa - \frac{\dot x}{y} < 0,$$

and

$$2Ha + 1 = a\kappa(0) \leqq 0.$$

Hence, an arbitrary (non-zero) constant mean curvature surface of revolution is obtained by one of the expression (4.2) for $H < 0$ and $2Ha + 1 \leqq 0$.

Here, a surface obtained by replacing (H, a, s) by $(-H, -a, -s)$ in the expression (4.2) is congruent to the original one. Hence we may assume $H > 0$ without loss of generality. In addition, the change

$$(H, a, s) \mapsto (H', a', s') = \left(H, -a - \frac{1}{H}, s + \frac{\pi}{2H}\right)$$

keeps the curve unchanged, and $2H'a' + 1 = -(2Ha+1)$. Thus, we may assume $2Ha + 1 \geqq 0$ without loss of generality. □

Special Cases. Let $H > 0$ be a positive constant.

Example 4.2 (The circular cylinder)**.** When $a = -1/(2H)$, (4.2) turns to be

$$x(s) = -s, \qquad y(s) = \frac{1}{2H}.$$

The corresponding surface is a circular cylinder of radius $1/(2H)$.

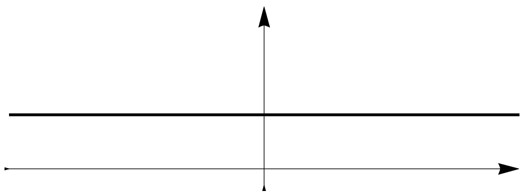

Figure 1: Example 4.2

Example 4.3 (The spheres)**.** When $a = 0$, (4.2) turns to be

$$\dot{x}(s) = -|\sin Hs|, \qquad y(s) = \frac{|\sin Hs|}{H}.$$

Integrating the first equation with respect to s, we have

$$x(s) = \frac{1}{H}\cos H\tau - \frac{n+1}{H}, \quad (Hs = n\pi + \tau, n \in \mathbb{Z}, \tau \in [0, \pi)).$$

The corresponding surface has singularities on $s = n\pi$ $(n \in \mathbb{Z})$, and its image is a sequence of infinitely many spheres with radius $1/H$ centered at $\frac{n\pi}{2H}$ $(n \in \mathbb{Z})$.

Generic Cases: Unduloids and Nodoids If $2Ha + 1 \notin \{0, 1\}$, $y(s)$ in (4.2) defined on $\mathbb{R}$ because

$$\begin{aligned}(2Ha+1)^2 &- 2(2Ha+1)\cos 2Hs + 1 \\ &\geqq (2Ha+1)^2 - 2(2Ha+1) + 1 = (2Ha)^2 > 0,\end{aligned}$$

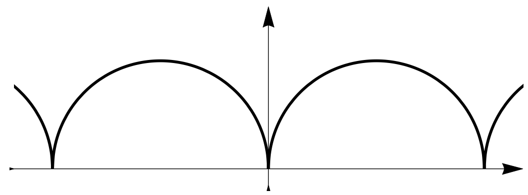

Figure 2: Example 4.3

and obviously $2\pi/H$-periodic. On the other hand, $x(s)$ in (4.2) has the following periodicity:

$$\begin{aligned} x\left(s + \frac{2\pi}{H}\right) &= x(s) + c, \\ c := \int_0^{2\pi/H} &\frac{(2Ha+1)\cos 2Hu - 1}{2Hy(u)}\,du. \end{aligned} \tag{4.3}$$

Remark that the integration in (4.3) cannot be expressed in terms of elementary functions. In fact, it is an elliptic integral.

Proposition 4.4. *Let $H > 0$ and $a \in (-1/(2H), 0)$ be constants. Then $x(s)$ in (4.2) is a decreasing function Sikh $\dot{x}(s) < 0$. Then the curve $\gamma(s) = (x(s), y(s))$ has no self-intersection, and can be expressed as the graph $y = f(x)$.*

Proof. Since $0 < 2Ha + 1 < 1$, $(2Ha + 1)\cos 2Hs - 1 < 0$ for each s. Hence

$$\dot{x}(s) = \frac{(2Ha+1)\cos 2Hs - 1}{2Hy(u)} < 0,$$

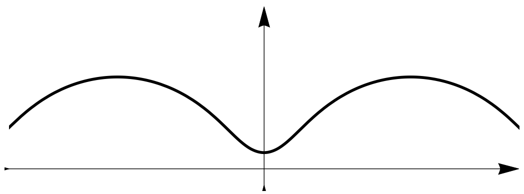

Figure 3: an unduloid

and the conclusion follows. □

A surface as in Proposition 4.4 is called an *unduloid* (Figure 3).

On the other hand, when $a > 0$, $x(s)$ is neither increasing nor decreasing. In fact,

Proposition 4.5. *Let $H > 0$ and $a > 0$ be constants. Then the curve $\gamma(s) = (x(s), y(s))$ in (4.2) have countably many self-intersections.*

A surface as in Proposition 4.5 is called an *nodoid* (Figure 4).

Plotting Delaunay surfaces. The profile curve of an unduloid is obtained as the locus of a focal point of an ellipse while rolling it without slippage along a given line (C. E. Delaunay), see [4-1] (Appendix B-6) and/or [4-2] (Appendix B-7). In fact, we show that such a surface has constant mean curvature: As a preliminary, we notice that

$$r = r(\theta) = \frac{a}{1 + e\cos\theta} \qquad (a > 0,\ 0 < e < 1) \tag{4.4}$$

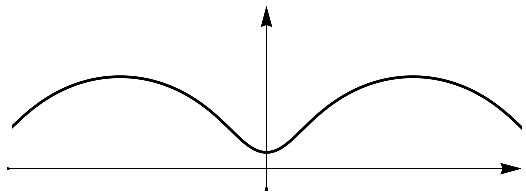

Figure 4: a nodoid

on the plane with respect to the polar coordinate system (r, θ) represents an ellipse such that O is one of the focal points, and e is its eccentricity. Since this ellipse can be parametrized as $\gamma(\theta) := r(\theta)(\cos\theta, \sin\theta)$, the tangent vector at $\mathrm{P} = \gamma(\theta)$ is computed as

$$\dot\gamma(\theta) \left(= \frac{d\gamma}{d\theta}\right) = \left(\frac{-a\sin\theta}{(1 + e\cos\theta)^2}, \frac{a(e + \cos\theta)}{(1 + e\cos\theta)^2}\right).$$

Let ξ be the angle between the vector $\overrightarrow{\mathrm{PO}}$ and the tangent of the ellipse at P (Fig. 5, left). Then we have

$$\cos\xi = \frac{-e\sin\theta}{\sqrt{1 + 2e\cos\theta + e^2}}, \qquad \sin\xi = \frac{1 + e\cos\theta}{\sqrt{1 + 2e\cos\theta + e^2}}.$$

We rotate the ellipse along the x-axis as in Fig. 5, right. When the ellipse has rotated angle θ about the focal point, then the point tangent to the x-axis has traveled the distance $s(\theta)$, which is the arc-length of $\gamma(\theta)$, that is,

$$s(\theta) = \int_0^\theta \frac{a\sqrt{1 + 2e\cos t + \varepsilon^2}}{(1 + e\cos t)^2}\, dt.$$

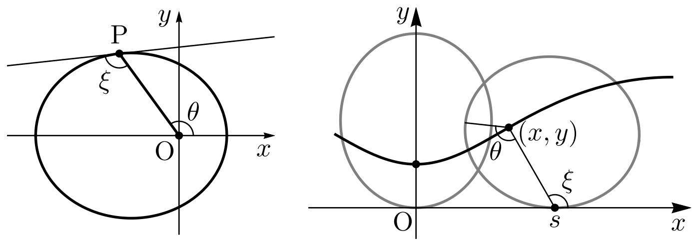

Figure 5:

Then the focal point of the ellipse is represented as

$$(x, y) = (x(\theta), y(\theta)) = (s(\theta) + r(\theta)\cos\xi(\theta), r(\theta)\sin\xi(\theta)).$$

Here, the mean curvature of the surface obtained by rotating the curve $(x(\theta), y(\theta))$ around the x-axis is computed as

$$H = \frac{\dot{y}\ddot{x} - \ddot{y}\dot{x}}{2(\sqrt{\dot{x}^2 + \dot{y}^2})^3} + \frac{\dot{x}}{2y\sqrt{\dot{x}^2 + \dot{y}^2}}.$$

Here,

$$\dot{x} = \frac{a(1 + e\cos\theta)}{\Delta^3}, \quad \dot{y} = \frac{ae\sin\theta}{\Delta^3}, \quad \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{a}{\Delta^2},$$

where $\Delta := \sqrt{1 + 2e\cos\theta + e^2}$. Hence we have $H = (1 - e^2)/(2a)$, which is a constant.

On the other hand, consider rotating a hyperbola along a line as in Fig. 6. Continuing the rotation, the tangent intersection of the hyperbola and the line moves out to infinity, and

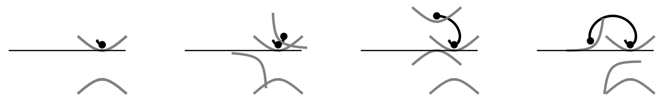

Figure 6:

the line tends to the asymptotic line of the hyperbola. From this limit state, we continue rotating the other component of the hyperbola along the given line. Repeating these over and over again, the locus of the focal point of the hyperbola is the generating curve of a nodoid. In fact, the polar equation of the ellipse (4.4), used in the case of an unduloid, also represents a hyperbola when $e > 1$. Then the mean curvature of the rotated surface can be computed similarly, which is constant.

References

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Exercises

4-1 Explain what is a surfaces of revolution obtained by the locus of a focal point of an parabola while rolling it without slippage along a given line?

The Laplacian

Riemannian 2-manifolds. Let Σ be a 2 dimensional manifold. A *Riemannian metric* ds^2 of Σ is a collections of (positive definite) inner product of the tangent space $T_p\Sigma$ of Σ at p, here p runs over whole Σ. Then, for each $p \in \Sigma$, $(ds^2)_p$ is an inner product of the vector space $T_p\Sigma$. Let $(U; u, v)$ be a local coordinate system of Σ, then $\{\partial/\partial u, \partial/\partial v\}$ is a field of bases on U, namely, $\{(\partial/\partial u)_p, (\partial/\partial v)_p\}$ is a basis of $T_p\Sigma$ for each $p \in U$. We write the matrix representation of ds^2 with respect to such a field of bases as

$$\widehat{I} := \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \qquad \text{where} \quad \begin{aligned} E &= \left(\tfrac{\partial}{\partial u}, \tfrac{\partial}{\partial u}\right), \\ F &= \left(\tfrac{\partial}{\partial u}, \tfrac{\partial}{\partial v}\right), \\ G &= \left(\tfrac{\partial}{\partial v}, \tfrac{\partial}{\partial v}\right). \end{aligned} \tag{5.1}$$

Here, $(\ ,\ )$ denotes the inner product induced by ds^2. The Riemannian metric ds^2 is said to be *smooth* if E, F and G in (5.1) are smooth functions in (u, v). Note that this condition is independent of a choice of coordinate system. Throughout this section, Riemannian metrics are assumed to be smooth. Under the situation as in (5.1), we write

$$ds^2 := E\,du^2 + 2F\,du\,dv + G\,dv^2. \tag{5.2}$$

Lemma 5.1. *Let ds^2 in (5.2) be a Riemannian metric. Then*

$$E > 0, \qquad G > 0, \qquad \text{and} \qquad EG - F^2 > 0$$

holds.

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Proof. Since ds^2 is positive definite,

$$(\boldsymbol{v}, \boldsymbol{v}) = Ea^2 + 2Fab + Gb^2 > 0$$

holds for an arbitrary

$$\boldsymbol{v} := a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}.$$

In particular, letting $(a, b) = (1, 0)$ and $(0, 1)$, we have $E, G > 0$. Moreover, when $(a, b) = (-F, E)$, it holds that

$$0 < EF^2 - 2F^2E + E^2G = E(EG - F^2).$$

Then we have the conclusion. □

Assume the manifold Σ is oriented, and take a coordinate system $(U; u, v)$ on Σ which is compatible of the orientation. We call the differential 2-form

$$dA := \sqrt{EG - F^2}\,du \wedge dv \tag{5.3}$$

the *area element*.

Lemma 5.2. *The area element* (5.3) *does not depend on a choice of coordinate system compatible to the orientation.*

Proof. Let $(V; \xi, \eta)$ be another coordinate system such that the intersection with $(U; u, v)$ is not empty. Then

$$\left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}\right) = \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) J, \qquad J := \begin{pmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} & \frac{\partial v}{\partial \eta} \end{pmatrix}, \tag{5.4}$$

here we call J the *Jacobian matrix* of the coordinate change $(\xi,\eta)\mapsto(u,v)$. If we write

$$ds^2 = \widetilde{E}\,d\xi^2 + 2\widetilde{F}\,d\xi\eta^2 + \widetilde{G}\,dv^2,$$

E, F, G as in (**??**) and $\widetilde{E}$, $\widetilde{F}$, $\widetilde{G}$ are related as

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = {}^tJ \begin{pmatrix} \widetilde{E} & \widetilde{F} \\ \widetilde{F} & \widetilde{G} \end{pmatrix} J. \tag{5.5}$$

On the other hand,

$$\begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = J \begin{pmatrix} du \\ dv \end{pmatrix}. \tag{5.6}$$

Noticing $\det J > 0$ because (u,v) and (ξ,η) are compatible to the orientation, the conclusion follows by these equalities. □

Example 5.3. Let Σ is an oriented 2-manifold and $f\colon\Sigma\to\mathbb{R}^3$ an immersion. Then, for each $p\in\Sigma$, the restriction canonical inner product "·" of $\mathbb{R}^3$ to $df(T_p\Sigma)\subset\mathbb{R}^3$ gives an inner product of $T_p\Sigma$, by identifying $T_p\Sigma$ and $df(T_p\Sigma)$. Thus, we have *the Riemannian metric ds^2 induced by the immersion f* which is nothing but the first fundamental form as in (1.2).

L^2-inner product for smooth functions. product Let (Σ, ds^2) be a Riemannian manifold, and assume that the manifold is oriented, for the sake of simplicity. We denote

$$\begin{aligned} &C^\infty(\Sigma) := \text{the set of smooth functions on } \Sigma,\\ &C^\infty_0(\Sigma) := \{\varphi\in C^\infty(\Sigma)\,;\,\operatorname{supp}\varphi\subset\Sigma \text{ is compact}\},\end{aligned} \tag{5.7}$$

where $\operatorname{supp} f = \overline{\{p\in\Sigma\,;\,f(p)\neq 0\}}$ is the *support* of f. Then $C^\infty_0(\Sigma)$ is a linear subspace of the vector space $C^\infty(\Sigma)$.

Definition 5.4. The L^2-inner product of $C^\infty_0(\Sigma)$ is defined as

$$\langle\varphi,\psi\rangle := \int_\Sigma \varphi\psi\,dA \qquad (\varphi,\psi\in C^\infty_0(\Sigma))$$

where dA is the area element as in (5.3).

Then $\langle\ ,\ \rangle$ is an inner product of $C^\infty_0(\Sigma)$.

L^2-inner product of one forms. We denote

$$\Lambda^1(\Sigma) := \text{the set of smooth 1-forms on } \Sigma.$$

On a local coordinate system $(U;u,v)$, α, $\beta\in\Lambda^1(\Sigma)$ are expressed as

$$\alpha = \alpha_1\,du + \alpha_2\,dv, \qquad \beta = \beta_1\,du + \beta_2\,dv.$$

Then by (5.5) and (5.6),

$$(\alpha,\beta) := (\alpha_1,\alpha_2)\begin{pmatrix} E & F \\ F & G\end{pmatrix}^{-1}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} \tag{5.8}$$

does not depend on a choice of coordinate system.

Definition 5.5. We denote

$$\Lambda^1_0(\Sigma) := \{\alpha\in\Lambda^1(\Sigma)\,;\,\operatorname{supp}\alpha\subset\Sigma \text{ is compact}\},$$

and define the *L^2-inner product* of one forms as

$$\langle\alpha,\beta\rangle := \int_\Sigma (\alpha,\beta)\,dA \qquad (\alpha,\beta\in\Lambda^1_0(\Sigma)).$$

Definition 5.6. For $\alpha \in \Lambda^1(\Sigma)$, we define

$$\delta\alpha := -\frac{1}{\sqrt{g}}\left[\left(\frac{G\alpha_1 - F\alpha_2}{\sqrt{g}}\right)_u + \left(\frac{-F\alpha_1 + E\alpha_2}{\sqrt{g}}\right)_v\right], \tag{5.9}$$

where $\alpha = \alpha_1\,du + \alpha_2\,dv$, E, F and G are as in (5.1), and $g := EG - F^2$.

Lemma 5.7. *The right-hand side of* (5.9) *does not depend on a choice of coordinate system.*

Proposition 5.8. *For $\varphi \in C^\infty(\Sigma)$ and $\alpha \in \Lambda^1_0(\Sigma)$, it holds that*

$$\langle \varphi, \delta\alpha \rangle = \langle d\varphi, \alpha \rangle \tag{5.10}$$

Proof. It is sufficient to show the equality when $\operatorname{supp} f$ and $\operatorname{supp}\alpha$ are contained in a local coordinate system $(U; u, v)$. In this case,

$$\begin{aligned}(d\varphi, \alpha)\,dA &= (d\varphi, \alpha)\sqrt{g}\,du \wedge dv \\ &= \frac{1}{\sqrt{g}}\left(\varphi_u(G\alpha_1 - F\alpha_2) + \varphi_v(-F\alpha_1 + E\alpha_2)\right) \\ &= d\omega + \delta\alpha\,dA\end{aligned}$$

hold for some one form ω, proving the conclusion. □

The Laplacian

Definition 5.9. The map $\Delta_{ds^2} \colon C^\infty(\Sigma) \to C^\infty(\Sigma)$ defined by

$$\Delta_{ds^2}\varphi := -\delta d\varphi \qquad (\varphi \in C^\infty(\Sigma))$$

is called the *Laplacian* with respect to the Riemannian metric ds^2.

Proposition 5.10. *For each φ, $\psi \in \Lambda^1_0(\Sigma)$,*

$$\int_\Sigma \varphi \Delta_{ds^2}\psi\,dA = -\langle d\varphi, d\psi \rangle$$

holds.

Proof. By Proposition 5.8,

$$\begin{aligned}\int_\Sigma \varphi \Delta_{ds^2}\psi\,dA &= \langle \varphi, \Delta_{ds^2}\psi \rangle = -\langle \varphi, \delta d\psi \rangle = -\langle d\varphi, d\psi \rangle \\ &= -\langle d\varphi, d\psi \rangle\end{aligned}$$

holds. □

A function $\varphi \in C^\infty(\Sigma)$ satisfying $\Delta_{ds^2}\varphi = 0$ is called a *harmonic function*.

Corollary 5.11. *A harmonic function on a compact, connected Riemannian manifold is a constant.*

Proof. Since Σ is compact, $C^\infty_0(\Sigma) = C^\infty(\Sigma)$. If φ is harmonic,

$$0 = \int_\Sigma \varphi \Delta_{ds^2}\varphi = -\langle d\varphi, d\varphi \rangle,$$

and hence $d\varphi = 0$. □

References

[5-1] 梅原雅顕・山田光太郎：曲線と曲面—微分幾何的アプローチ（改訂版），裳華房，2014.

Exercises

5-1[H] Consider the situation in Example 5.3, that is, $f\colon \Sigma \to \mathbb{R}^3$ be an immersion with the first fundamental form ds^2.

We write $f = (f_1, f_2, f_3)$, where f_j's $(j = 1, 2, 3)$ are smooth functions defined on Σ. Then

$$\varDelta_{ds^2} f := (\varDelta_{ds^2} f_1, \varDelta_{ds^2} f_2, \varDelta_{ds^2} f_3)$$

is a vector valued function defined on Σ.

(1) Let $(U; u, v)$ be a local coordinate system of Σ. Show that $\varDelta_{ds^2} f$ is perpendicular to both f_u and f_v.

(2) Show that

$$\varDelta_{ds^2} f = 2H\nu,$$

where H and ν are the mean curvature and the unit normal vector field, respectively.

(3) An immersion f is said to be *minimal* if the mean curvature vanishes identically (see Definition 1.2). Prove that there are no compact minimal surface without boundary.

5-2[H] Let (Σ, ds^2) be a Riemannian 2-manifold. A coordinate system $(U; u, v)$ is said to be *isothermal* or *conformal* if the metric ds^2 is written as

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \tag{5.11}$$

where σ is a smooth function defined on U.

(1) Compute $\varDelta_{ds^2}\varphi$ with respect to the coordinate system (u, v).

(2) Let $(V; \xi, \eta)$ and $(U; u, v)$ are isothermal coordinate systems. Then the coordinate change

$$(\xi, \eta) \longmapsto (u(\xi, \eta), v(\xi, \eta))$$

satisfy

$$\varDelta_{ds^2} u = \varDelta_{ds^2} v = 0,$$

that is, coordinate changes between isothermal coordinate systems are harmonic.