

Linear Ordinary Differential Equations

Preliminaries: Matrix Norms. Denote by $M_n(\mathbb{R})$ the set of $n \times n$ matrix with real components, which can be identified the vector space \mathbb{R}^{n^2} . In particular, the Euclidean norm of \mathbb{R}^{n^2} induces a norm

$$(1.1) \quad |X|_{\mathbb{E}} = \sqrt{\operatorname{tr}({}^tXX)} = \sqrt{\sum_{i,j=1}^n x_{ij}^2}$$

on $M_n(\mathbb{R})$. On the other hand, we let

$$(1.2) \quad |X|_{\mathbb{M}} := \sup \left\{ \frac{|X\mathbf{v}|}{|\mathbf{v}|}; \mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\} \right\},$$

where $|\cdot|$ on the right-hand side denotes the Euclidean norm of \mathbb{R}^n .

Lemma 1.1. (1) *The map $X \mapsto |X|_{\mathbb{M}}$ is a norm of $M_n(\mathbb{R})$.*

(2) *For $X, Y \in M_n(\mathbb{R})$, it holds that $|XY|_{\mathbb{M}} \leq |X|_{\mathbb{M}}|Y|_{\mathbb{M}}$.*

(3) *Let $\lambda = \lambda(X)$ be the maximum eigenvalue of semi-positive definite symmetric matrix tXX . Then $|X|_{\mathbb{M}} = \sqrt{\lambda}$ holds.*

(4) $(1/\sqrt{n})|X|_{\mathbb{E}} \leq |X|_{\mathbb{M}} \leq |X|_{\mathbb{E}}$.

(5) *The map $|\cdot|_{\mathbb{M}}: M_n(\mathbb{R}) \rightarrow \mathbb{R}$ is continuous with respect to the Euclidean norm.*

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Proof. Since $|X\mathbf{v}|/|\mathbf{v}|$ is invariant under scalar multiplications to \mathbf{v} , we have $|X|_{\mathbb{M}} = \sup\{|X\mathbf{v}|; \mathbf{v} \in S^{n-1}\}$, where S^{n-1} is the unit sphere in \mathbb{R}^n . Here, the $S^{n-1} \ni \mathbf{x} \mapsto |A\mathbf{x}| \in \mathbb{R}$ is a continuous function defined on a compact space, and so the map takes maximum. Thus, the right-hand side of (1.2) is well-defined. It is easy to verify that $|\cdot|_{\mathbb{M}}$ satisfies the axiom of the norm.

Since $A := {}^tXX$ is positive semi-definite the eigenvalues λ_j ($j = 1, \dots, n$) are non-negative real numbers. In particular, there exists an orthonormal basis $[\mathbf{a}_j]$ of \mathbb{R}^n satisfying $A\mathbf{a}_j = \lambda_j\mathbf{a}_j$ ($j = 1, \dots, n$). Let λ be the maximum eigenvalues of A , and write $\mathbf{v} = v_1\mathbf{a}_1 + \dots + v_n\mathbf{a}_n$. Then it holds that

$$\langle X\mathbf{v}, X\mathbf{v} \rangle = \lambda_1 v_1^2 + \dots + \lambda_n v_n^2 \leq \lambda \langle \mathbf{v}, \mathbf{v} \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product of \mathbb{R}^n . The equality of this inequality holds if and only if \mathbf{v} is the λ -eigenvector, proving (3). Noticing the norm (1.1) is invariant under conjugations $X \mapsto {}^tPXP$ ($P \in O(n)$), we obtain $|X|_{\mathbb{E}} = \sqrt{\lambda_1 + \dots + \lambda_n}$ by diagonalizing tXX by an orthogonal matrix P . Then we obtain (4). Hence two norms $|\cdot|_{\mathbb{E}}$ and $|\cdot|_{\mathbb{M}}$ induce the same topology $M_n(\mathbb{R})$. In particular, we have (5). \square

Preliminaries: Matrix-valued Functions.

Lemma 1.2. *Let X and Y be C^∞ -maps defined on a domain $U \subset \mathbb{R}^m$ into $M_n(\mathbb{R})$. Then*

$$(1) \quad \frac{\partial}{\partial u_j}(XY) = \frac{\partial X}{\partial u_j}Y + X\frac{\partial Y}{\partial u_j},$$

$$(2) \quad \frac{\partial}{\partial u_j} \det X = \operatorname{tr} \left(\tilde{X} \frac{\partial X}{\partial u_j} \right), \text{ and}$$

$$(3) \quad \frac{\partial}{\partial u_j} X^{-1} = -X^{-1} \frac{\partial X}{\partial u_j} X^{-1},$$

where \tilde{X} is the cofactor matrix of X , and we assume in (3).

Proposition 1.3. Assume two C^∞ matrix-valued functions $X(t)$ and $\Omega(t)$ satisfy

$$(1.3) \quad \frac{dX(t)}{dt} = X(t)\Omega(t), \quad X(t_0) = X_0.$$

Then

$$(1.4) \quad \det X(t) = (\det X_0) \exp \int_{t_0}^t \operatorname{tr} \Omega(\tau) d\tau$$

holds. In particular, if $X_0 \in \operatorname{GL}(n, \mathbb{R})$,¹ then $X(t) \in \operatorname{GL}(n, \mathbb{R})$ for all t .

Proof. By (2) of Lemma 1.2, we have

$$\begin{aligned} \frac{d}{dt} \det X(t) &= \operatorname{tr} \left(\tilde{X}(t) \frac{dX(t)}{dt} \right) = \operatorname{tr} \left(\tilde{X}(t) X(t) \Omega(t) \right) \\ &= \operatorname{tr} (\det X(t) \Omega(t)) = \det X(t) \operatorname{tr} \Omega(t). \end{aligned}$$

Here, we used the relation $\tilde{X}X = X\tilde{X} = (\det X) \operatorname{id}^2$. Hence $\frac{d}{dt} (\rho(t)^{-1} \det X(t)) = 0$, where $\rho(t)$ is the right-hand side of (1.4). \square

¹ $\operatorname{GL}(n, \mathbb{R}) = \{A \in \operatorname{M}_n(\mathbb{R}); \det A \neq 0\}$: the general linear group.

²In this lecture, id denotes the identity matrix.

Proposition 1.4. Assume $\Omega(t)$ in (1.3) is skew-symmetric for all t , that is, ${}^t\Omega + \Omega$ is identically O . If $X_0 \in \operatorname{O}(n)$ (resp. $X_0 \in \operatorname{SO}(n)$)³, $X(t) \in \operatorname{O}(n)$ (resp. $X(t) \in \operatorname{SO}(n)$) for all t .

Proof. By (1) in Lemma 1.2,

$$\begin{aligned} \frac{d}{dt} (X^t X) &= \frac{dX}{dt} {}^t X + X \left(\frac{dX}{dt} \right) \\ &= X \Omega^t X + X^t \Omega^t X = X(\Omega + {}^t\Omega)^t X = O. \end{aligned}$$

Hence $X^t X$ is constant, that is, if $X_0 \in \operatorname{O}(n)$,

$$X(t)^t X(t) = X(t_0)^t X(t_0) = X_0^t X_0 = \operatorname{id}.$$

if $X_0 \in \operatorname{O}(n)$, proves the first case of the proposition. Since $\det A = \pm 1$ when $A \in \operatorname{O}(n)$, the second case follows by continuity of $\det X(t)$. \square

Preliminaries: Norms of Matrix-Valued functions. Let $I = [a, b]$ be a closed interval, and denote by $C^0(I, \operatorname{M}_n(\mathbb{R}))$ the set of continuous functions $X: I \rightarrow \operatorname{M}_n(\mathbb{R})$. For any fixed number k , we define

$$(1.5) \quad \|X\|_{I,k} := \sup \{ e^{-kt} |X(t)|_{\operatorname{M}}; t \in I \}$$

for $X \in C^0(I, \operatorname{M}_n(\mathbb{R}))$. When $k = 0$, $\|\cdot\|_{I,0}$ is the uniform norm for continuous functions, which is complete. Similarly, one can prove the following in the same way:

³ $\operatorname{O}(n) = \{A \in \operatorname{M}_n(\mathbb{R}); {}^t A A = A^t A = \operatorname{id}\}$: the orthogonal group; $\operatorname{SO}(n) = \{A \in \operatorname{O}(n); \det A = 1\}$: the special orthogonal group.

Lemma 1.5. *The map $\|\cdot\|_{J,k}: C^0(I, M_n(\mathbb{R}))$ is a complete norm.*

Linear Ordinary Differential Equations. We prove the fundamental theorem for *linear* ordinary differential equations.

Proposition 1.6. *Let $\Omega(t)$ be a C^∞ -function valued in $M_n(\mathbb{R})$ defined on an interval I . Then for each $t_0 \in I$, there exists the unique matrix-valued C^∞ -function $X(t) = X_{t_0, \text{id}}(t)$ such that*

$$(1.6) \quad \frac{dX(t)}{dt} = X(t)\Omega(t), \quad X(t_0) = \text{id}.$$

Proof. Uniqueness: Assume $X(t)$ and $Y(t)$ satisfy (1.6). Then

$$\begin{aligned} Y(t) - X(t) &= \int_{t_0}^t (Y'(\tau) - X'(\tau)) d\tau \\ &= \int_{t_0}^t (Y(\tau) - X(\tau))\Omega(\tau) d\tau \end{aligned}$$

holds. Hence for an arbitrary closed interval $J \subset I$,

$$\begin{aligned} |Y(t) - X(t)|_{\mathbb{M}} &\leq \left| \int_{t_0}^t |(Y(\tau) - X(\tau))\Omega(\tau)|_{\mathbb{M}} d\tau \right| \\ &\leq \left| \int_{t_0}^t |Y(\tau) - X(\tau)|_{\mathbb{M}} |\Omega(\tau)|_{\mathbb{M}} d\tau \right| \\ &= \left| \int_{t_0}^t e^{-k\tau} |Y(\tau) - X(\tau)|_{\mathbb{M}} e^{k\tau} |\Omega(\tau)|_{\mathbb{M}} d\tau \right| \end{aligned}$$

$$\begin{aligned} &\leq \|Y - X\|_{J,k} \sup_J |\Omega|_{\mathbb{M}} \left| \int_{t_0}^t e^{k\tau} d\tau \right| \\ &= \|Y - X\|_{J,k} \frac{\sup_J |\Omega|_{\mathbb{M}}}{|k|} e^{kt} \left| 1 - e^{-k(t-t_0)} \right| \end{aligned}$$

holds for $t \in J$. Here, setting $J = [t_0, a]$ and $k = 2 \sup_J |\Omega|_{\mathbb{M}}$, we have

$$\|Y - X\|_{J,k} \leq \frac{1}{2} \|Y - X\|_{J,k},$$

that is, $\|Y - X\|_{J,k} = 0$, proving $Y(t) = X(t)$ for $t \in J$. Similarly, on the interval $J' = [a, t_0]$, we can conclude $Y = X$ on J' setting $k = -2 \sup_J |\Omega|_{\mathbb{M}}$. Since J and J' are arbitrary, $Y = X$ holds on I .

Existence: Let $J := [t_0, a] \subset I$ be a closed interval, and define a sequence $\{X_j\}$ of matrix-valued functions defined on I satisfying $X_0(t) = \text{id}$ and

$$(1.7) \quad X_{j+1}(t) = \text{id} + \int_{t_0}^t X_j(\tau)\Omega(\tau) d\tau \quad (j = 0, 1, 2, \dots).$$

Let $k := 2 \sup_J |\Omega|_{\mathbb{M}}$. Then

$$\begin{aligned} |X_{j+1}(t) - X_j(t)|_{\mathbb{M}} &\leq \int_{t_0}^t |X_j(\tau) - X_{j-1}(\tau)|_{\mathbb{M}} |\Omega(\tau)|_{\mathbb{M}} d\tau \\ &\leq e^{kt} \|X_j - X_{j-1}\|_{J,k} \frac{\sup_J |\Omega|_{\mathbb{M}}}{k} = \frac{e^{kt}}{2} \|X_j - X_{j-1}\|_{J,k}, \end{aligned}$$

and hence $\|X_{j+1} - X_j\|_{J,k} \leq \frac{1}{2} \|X_j - X_{j-1}\|_{J,k}$, that is, $\{X_j\}$ is a Cauchy sequence with respect to $\|\cdot\|_{J,k}$. Thus, by completeness

(Lemma 1.5), it converges to some $X \in C^0(J, M_n(\mathbb{R}))$. By (1.7), the limit X satisfies

$$X(t_0) = \text{id}, \quad X(t) = \text{id} + \int_{t_0}^t X(\tau)\Omega(\tau) d\tau.$$

Applying the fundamental theorem of calculus, we can see that X satisfies $X'(t) = X(t)\Omega(t)$ ($' = d/dt$). Since J can be taken arbitrarily, existence of the solution on $I \cap \{t \geq t_0\}$ is proved. Existence of $I \cap \{t \leq t_0\}$ can be proved in the same way. So far, existence of a differentiable function $X(t)$ satisfying (1.6) is obtained.

Finally, we shall prove that X is of class C^∞ . Since $X'(t) = X(t)\Omega(t)$, the derivative X' of X is continuous. Hence X is of class C^1 , and so is $X(t)\Omega(t)$. Thus we have that $X'(t)$ is of class C^1 , and then X is of class C^2 . Iterating this argument, we can prove that $X(t)$ is of class C^r for arbitrary r . \square

Corollary 1.7. *Let $\Omega(t)$ be a matrix-valued C^∞ -function defined on an interval I . Then for each $t_0 \in I$ and $X_0 \in M_n(\mathbb{R})$, there exists the unique matrix-valued C^∞ -function $X(t) = X_{t_0, X_0}(t)$ defined on I such that*

$$(1.8) \quad \frac{dX(t)}{dt} = X(t)\Omega(t), \quad X(t_0) = X_0.$$

In particular, $X_{t_0, X_0}(t)$ is of class C^∞ in X_0 and t .

Proof. We rewrite $X(t)$ in Proposition 1.6 as $Y(t) = X_{t_0, \text{id}}(t)$. Then the function

$$(1.9) \quad X(t) := X_0 Y(t) = X_0 X_{t_0, \text{id}}(t),$$

is desired one. Conversely, assume $X(t)$ satisfies the conclusion. Noticing $Y(t)$ is a regular matrix for all t because of Proposition 1.3,

$$W(t) := X(t)Y(t)^{-1}$$

satisfies

$$\begin{aligned} \frac{dW}{dt} &= \frac{dX}{dt}Y^{-1} - X(t)Y^{-1}\frac{dY}{dt}Y^{-1} \\ &= X\Omega Y^{-1} - XY^{-1}Y\Omega Y^{-1} = O. \end{aligned}$$

Hence

$$W(t) = W(t_0) = X(t_0)Y(t_0)^{-1} = X_0.$$

Hence the uniqueness is obtained. The final part is obvious by the expression (1.9). \square

Proposition 1.8. *Let $\Omega(t)$ and $B(t)$ be a matrix-valued C^∞ -functions defined on I . Then for each $t_0 \in I$ and $X_0 \in M_n(\mathbb{R})$, there exists the unique matrix-valued C^∞ -function defined on I satisfying*

$$(1.10) \quad \frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \quad X(t_0) = X_0.$$

Proof. Rewrite X in Proposition 1.6 as $Y(t) := X_{t_0, \text{id}}(t)$. Then

$$(1.11) \quad X(t) = \left(X_0 + \int_{t_0}^t B(\tau)Y^{-1}(\tau) d\tau \right) Y(t)$$

satisfies (1.10). Conversely, if X satisfies (1.10), $W := XY^{-1}$ satisfies

$$X' = W'Y + WY' = W'Y + WY\Omega, \quad X\Omega + B = WY\Omega + B,$$

and then we have $W' = BY^{-1}$. Since $W(t_0) = X_0$,

$$W = X_0 + \int_{t_0}^t B(\tau)Y^{-1}(\tau) d\tau.$$

Thus we obtain (1.11). \square

Theorem 1.9. *Let I and U be an interval and a domain in \mathbb{R}^m , respectively, and let $\Omega(t, \boldsymbol{\alpha})$ and $B(t, \boldsymbol{\alpha})$ be matrix-valued C^∞ -functions defined on $I \times U$ ($\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$). Then for each $t_0 \in I$, $\boldsymbol{\alpha} \in U$ and $X_0 \in M_n(\mathbb{R})$, there exists the unique matrix-valued C^∞ -function $X(t) = X_{t_0, X_0, \boldsymbol{\alpha}}(t)$ defined on I such that*

$$(1.12) \quad \frac{dX(t)}{dt} = X(t)\Omega(t, \boldsymbol{\alpha}) + B(t, \boldsymbol{\alpha}), \quad X(t_0) = X_0.$$

Moreover,

$$I \times I \times M_n(\mathbb{R}) \times U \ni (t, t_0, X_0, \boldsymbol{\alpha}) \mapsto X_{t_0, X_0, \boldsymbol{\alpha}}(t) \in M_n(\mathbb{R})$$

is C^∞ -map.

Proof. Let $\tilde{\Omega}(t, \tilde{\boldsymbol{\alpha}}) := \Omega(t + t_0, \boldsymbol{\alpha})$ and $\tilde{B}(t, \tilde{\boldsymbol{\alpha}}) = B(t + t_0, \boldsymbol{\alpha})$, and let $\tilde{X}(t) := X(t + t_0)$. Then (1.12) is equivalent to

$$(1.13) \quad \frac{d\tilde{X}(t)}{dt} = \tilde{X}(t)\tilde{\Omega}(t, \tilde{\boldsymbol{\alpha}}) + \tilde{B}(t, \tilde{\boldsymbol{\alpha}}), \quad \tilde{X}(0) = X_0,$$

where $\tilde{\boldsymbol{\alpha}} := (t_0, \alpha_1, \dots, \alpha_m)$. There exists the unique solution $\tilde{X}(t) = \tilde{X}_{\text{id}, X_0, \tilde{\boldsymbol{\alpha}}}(t)$ of (1.13) for each $\tilde{\boldsymbol{\alpha}}$ because of Proposition 1.8. So it is sufficient to show differentiability with respect

to the parameter $\tilde{\boldsymbol{\alpha}}$. We set $Z = Z(t)$ as the unique solution of

$$(1.14) \quad \frac{dZ}{dt} = Z\tilde{\Omega} + \tilde{X} \frac{\partial \tilde{\Omega}}{\partial \alpha_j} + \frac{\partial \tilde{B}}{\partial \alpha_j}, \quad Z(0) = O.$$

Then it holds that $Z = \partial \tilde{X} / \partial \alpha_j$ (Problem 1-1). In particular, by the proof of Proposition 1.8, it holds that

$$Z = \frac{\partial \tilde{X}}{\partial \alpha_j} = \left(\int_0^t \left(\tilde{X}(\tau) \frac{\partial \tilde{\Omega}(\tau, \tilde{\boldsymbol{\alpha}})}{\partial \alpha_j} + \frac{\partial \tilde{B}(\tau, \tilde{\boldsymbol{\alpha}})}{\partial \alpha_j} \right) Y^{-1}(\tau) d\tau \right) Y(t).$$

Here, $Y(t)$ is the unique matrix-valued C^∞ -function satisfying $Y'(t) = Y(t)\tilde{\Omega}(t, \tilde{\boldsymbol{\alpha}})$, and $Y(0) = \text{id}$. Hence \tilde{X} is a C^∞ -function in $(t, \tilde{\boldsymbol{\alpha}})$. \square

Fundamental Theorem for Space Curves. As an application, we prove the fundamental theorem for space curves. A C^∞ -map $\gamma: I \rightarrow \mathbb{R}^3$ defined on an interval $I \subset \mathbb{R}$ into \mathbb{R}^3 is said to be a *regular curve* if $\dot{\gamma} \neq \mathbf{0}$ holds on I . For a regular curve $\gamma(t)$, there exists a parameter change $t = t(s)$ such that $\tilde{\gamma}(s) := \gamma(t(s))$ satisfies $|\tilde{\gamma}'(s)| = 1$. Such a parameter s is called the *arc-length parameter*.

Let $\gamma(s)$ be a regular curve in \mathbb{R}^3 parametrized by the arc-length satisfying $\gamma''(s) \neq \mathbf{0}$ for all s . Then

$$\mathbf{e}(s) := \gamma'(s), \quad \mathbf{n}(s) := \frac{\gamma''(s)}{|\gamma''(s)|}, \quad \mathbf{b}(s) := \mathbf{e}(s) \times \mathbf{n}(s)$$

forms a positively oriented orthonormal basis $\{\mathbf{e}, \mathbf{n}, \mathbf{b}\}$ of \mathbb{R}^3 for each s . Regarding each vector as column vector, we have the

matrix-valued function

$$(1.15) \quad \mathcal{F}(s) := (\mathbf{e}(s), \mathbf{n}(s), \mathbf{b}(s)) \in \text{SO}(3).$$

in s , which is called the *Frenet frame* associated to the curve γ . Under the situation above, we set

$$\kappa(s) := |\gamma''(s)| > 0, \quad \tau(s) := -\langle \mathbf{b}'(s), \mathbf{n}(s) \rangle,$$

which is called the *curvature* and *torsion*, respectively, of γ . Using these quantities, the Frenet frame satisfies

$$(1.16) \quad \frac{d\mathcal{F}}{ds} = \mathcal{F}\Omega, \quad \Omega = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

Proposition 1.10. *The curvature and the torsion are invariant under the transformation $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ of \mathbb{R}^3 ($A \in \text{SO}(3)$, $\mathbf{b} \in \mathbb{R}^3$). Conversely, two curves $\gamma_1(s)$, $\gamma_2(s)$ parametrized by arc-length parameter have common curvature and torsion, there exist $A \in \text{SO}(3)$ and $\mathbf{b} \in \mathbb{R}^3$ such that $\gamma_2 = A\gamma_1 + \mathbf{b}$.*

Proof. Let κ , τ and \mathcal{F}_1 be the curvature, torsion and the Frenet frame of γ_1 , respectively. Then the Frenet frame of $\gamma_2 = A\gamma_1 + \mathbf{b}$ ($A \in \text{SO}(3)$, $\mathbf{b} \in \mathbb{R}^3$) is $\mathcal{F}_2 = A\mathcal{F}_1$. Hence both \mathcal{F}_1 and \mathcal{F}_2 satisfy (1.16), and then γ_1 and γ_2 have common curvature and torsion.

Conversely, assume γ_1 and γ_2 have common curvature and torsion. Then the frenet frame \mathcal{F}_1 , \mathcal{F}_2 both satisfy (1.16). Let \mathcal{F} be the unique solution of (1.16) with $\mathcal{F}(t_0) = \text{id}$. Then by the proof of Corollary 1.7, we have $\mathcal{F}_j(t) = \mathcal{F}_j(t_0)\mathcal{F}(t)$ ($j = 1, 2$). In particular, since $\mathcal{F}_j \in \text{SO}(3)$, $\mathcal{F}_2(t) = A\mathcal{F}_1(t)$ ($A :=$

$\mathcal{F}_2(t_0)\mathcal{F}_1(t_0)^{-1} \in \text{SO}(3)$). Comparing the first column of these, $\gamma_2'(s) = A\gamma_1'(t)$ holds. Integrating this, the conclusion follows. \square

Theorem 1.11 (The fundamental theorem for space curves). *For given C^∞ -functions $\kappa(s)$ and $\tau(s)$ defined on I such that $\kappa(s) > 0$ on I . Then there exists a space curve $\gamma(s)$ parametrized by arc-length whose curvature and torsion are κ and τ , respectively. Moreover, such a curve is unique up to transformation $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ ($A \in \text{SO}(3)$, $\mathbf{b} \in \mathbb{R}^3$) of \mathbb{R}^3 .*

Proof. We have already shown the uniqueness in Proposition 1.10. We shall prove the existence: Let $\Omega(s)$ be as in (1.16), and $\mathcal{F}(s)$ the solution of (1.16) with $\mathcal{F}(s_0) = \text{id}$. Since Ω is skew-symmetric, $\mathcal{F}(s) \in \text{SO}(3)$ by Proposition 1.4. Denoting the column vectors of \mathcal{F} by \mathbf{e} , \mathbf{n} , \mathbf{b} , and let

$$\gamma(s) := \int_{s_0}^s \mathbf{e}(\sigma) d\sigma.$$

Then \mathcal{F} is the frenet frame of γ , and κ , and τ are the curvature and torsion of γ , respectively (Problem 1-2). \square

Exercises

1-1 Verify that Z in (1.14) coincides with $\partial\tilde{X}/\partial\alpha_j$.

1-2 Complete the proof of Theorem 1.11.