Isothermal parameters

A Review of Complex Analysis. Let $\mathbb{C}$ be the complex plane. A C^1-function[7] $f\colon \mathbb{C} \ni D \in z \mapsto w = f(z) \in \mathbb{C}$ defined on a domain D is said to be *holomorphic* if the derivative

$$f'(z) := \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists for all $z \in D$.

Fact 3.1 (The Cauchy-Riemann equation)**.** *A function $f\colon \mathbb{C} \ni D \to \mathbb{C}$ is holomorphic if and only if*

$$\frac{\partial u}{\partial \xi} = \frac{\partial v}{\partial \eta} \qquad \textit{and} \qquad \frac{\partial u}{\partial \eta} = -\frac{\partial v}{\partial \xi} \tag{3.1}$$

holds on D, where $w = f(z)$, $z = \xi + i\eta$, $w = u + iv$ ($i = \sqrt{-1}$).

For functions of complex variable $z = \xi + i\eta$, we set

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial \xi} - i\frac{\partial}{\partial \eta}\right), \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial \xi} + i\frac{\partial}{\partial \eta}\right). \tag{3.2}$$

Corollary 3.2. *For a complex function f,* (3.1) *is equivalent to*

$$\frac{\partial f}{\partial \bar{z}} = 0. \tag{3.3}$$

Proof. Setting $w = f(z) = u + iv$ and $z = \xi + i\eta$. Then the real (resp. imaginary) part of the left-hand side of (3.3) coincides with the first (resp. second) equation of (3.1). $\square$

26. June, 2018. Revised: 03. July, 2018

[7]Of class C^1 as a map from $D \subset \mathbb{R}^2$ to $\mathbb{R}^2$.

Isothermal Coordinates.

Definition 3.3. Let $f\colon M^2 \to \mathbb{R}^3$ be an immersion of 2-manifold, and ds^2 its first fundamental form. A local coordinate chart $\bigl(U;(u,v)\bigr)$ of M^2 is called an *isothermal coordinate system* or a *conformal coordinate system* if ds^2 is written in the form[8]

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \qquad \sigma = \sigma(u,v) \in C^\infty(U).$$

Example 3.4. Let $\gamma(u) = (x(u), z(u)) = (a\cosh\frac{u}{a}, u)$, that is, γ is the graph $x = a\cosh\frac{z}{a}$ on the xz-plane, called the *catenary*. We call the surface of revolution generated by $\gamma(u)$ the *catenoid*, which is parametrized as

$$p(u,v) = \bigl(x(u)\cos v, x(u)\sin v, z(u)\bigr),$$

This parametrization of the catenoid is isothermal when $a = 1$. In fact, the first fundamental form is expressed as $\cosh^2(u/a)(du^2 + a^2dv^2)$.

Definition 3.5. Two charts $\bigl(U_j;(u_j,v_j)\bigr)$ ($j = 1,2$) of a 2-manifold M^2 has the *same* (resp. *opposite*) *orientation* if the Jacobian $\frac{\partial(u_2,v_2)}{\partial(u_1,v_1)}$ is positive (resp. negative) on $U_1 \cap U_2$. A manifold M^2 is said to be *oriented* if there exists an atlas $\bigl\{\bigl(U_j;(u_j,v_j)\bigr)\bigr\}$ such that all charts have the same orientation. A choice of such an atlas is called an *orientation* of M^2.

[8]The notion of the isothermal coordinate system can be defined not only for surfaces but also for Riemannian 2-manifolds, that is, differentiable 2-manifolds M^2 with Riemannian metrics ds^2 (the first fundamental forms).

Proposition 3.6. *Let (u,v) be an isothermal coordinate system of a surface. Then another coordinate system (ξ,η) is also isothermal if and only if the parameter change $(\xi,\eta)\mapsto(u,v)$ satisfy*

$$\frac{\partial u}{\partial \xi}=\varepsilon\frac{\partial v}{\partial \eta},\qquad \frac{\partial u}{\partial \eta}=-\varepsilon\frac{\partial v}{\partial \xi}, \tag{3.4}$$

where $\varepsilon=1$ (resp. -1) if (u,v) and (ξ,η) has the same (resp. the opposite) orientation.

Proof. If we write $ds^2=e^{2\sigma}(du^2+dv^2)$, it holds that

$$ds^2=e^{2\sigma}\bigl((u_\xi^2+v_\xi^2)\,d\xi^2+2(u_\xi u_\eta+v_\xi v_\eta)\,d\xi\,d\eta+(u_\eta^2+v_\eta^2)\,d\eta^2\bigr).$$

Thus, (ξ,η) is isothermal if and only if

$$u_\xi^2+v_\xi^2=u_\eta^2+v_\eta^2,\qquad u_\xi u_\eta+v_\xi v_\eta=0. \tag{3.5}$$

The second equality yields $(u_\eta,v_\eta)=\varepsilon(-v_\xi,u_\xi)$ for some function ε. Substituting this into the first equation of (3.5), we get $\varepsilon=\pm1$. Moreover,

$$\frac{\partial(u,v)}{\partial(\xi,\eta)}=\det\begin{pmatrix}u_\xi & u_\eta\\ v_\xi & v_\eta\end{pmatrix}=\det\begin{pmatrix}u_\xi & -\varepsilon v_\xi\\ v_\xi & \varepsilon u_\xi\end{pmatrix}=\varepsilon(u_\xi^2+u_\eta^2).$$

Thus, the conclusion follows. □

Corollary 3.7. *Let (u,v) is an isothermal coordinate system. Then a coordinate system (ξ,η) is isothermal and has the same orientation as (u,v) if and only if the map $\xi+i\eta\mapsto u+iv$ $(i=\sqrt{-1})$ is holomorphic.*

Proof. Equations (3.4) for $\varepsilon=+1$ are nothing but the Cauchy-Riemann equations (3.1). □

The notion of isothermal coordinate systems are meaningful not only for immersed surfaces but also for Riemannian manifolds. There exist such coordinate systems on a 2-dimensional Riemannian manifold:

Fact 3.8 (Section 15 in 3-1)**.** *Let (M^2,ds^2) be an arbitrary Riemannian manifold. Then for each $p\in M^2$, there exists an isothermal chart containing p.*

Corollary 3.9. *Any oriented Riemannian 2-manifold has a structure of Riemann surface (i.e., a complex 1-manifold) such that for each complex coordinate $z=u+iv$, (u,v) is an isothermal coordinate system for the Riemannian metric.*

Proof. Let $p\in M^2$ and take a local coordinate chart $\bigl(U_p;(x,y)\bigr)$ at p which is compatible to the orientation of M^2. Then there exists an isothermal coordinate chart $\bigl(V_p;(u_p,v_p)\bigr)$ at p, because of Fact 3.8. Moreover, replacing (u,v) by (v,u) if necessary, we can take (u,v) which has the same orientation of (x,y). Thus, we have an atlas $\bigl\{\bigl(V_p;(u_p,v_p)\bigr)\bigr\}$ consisting of isothermal coordinate systems. Since each chart is compatible to the orientation, the coordinate change $z_p=u_p+iv_p\mapsto u_q+iv_q=z_q$ is holomorphic. Hence we get a complex atlas $\bigl\{\bigl(V_p;z_p\bigr)\bigr\}$. □

The Gauss and Weingarten formulas. Let $p\colon U\to\mathbb{R}^3$ be a parametrized regular surface defined on a domain U of the uv-plane. Assume that (u,v) is an isothermal coordinate system, and write the first fundamental form ds^2 as

$$ds^2 := e^{2\sigma}(du^2 + dv^2) \qquad \sigma \in C^\infty(U), \tag{3.6}$$

that is,

$$p_u \cdot p_u = p_v \cdot p_v = e^{2\sigma}, \qquad p_u \cdot p_v = 0, \tag{3.7}$$

where "$\cdot$" denotes the canonical inner product of $\mathbb{R}^3$. Since

$$|p_u \times p_v| = \sqrt{(p_u \cdot p_u)(p_v \cdot p_v) - (p_u \cdot p_v)^2} = e^{2\sigma},$$

the unit normal vector field ν can be chosen as

$$\nu = e^{-2\sigma}(p_u \times p_v), \tag{3.8}$$

where "$\times$" denotes the vector product of $\mathbb{R}^3$. Write the second fundamental form of p as

$$II = L\,du^2 + 2M\,du\,dv + N\,dv^2, \tag{3.9}$$

where

$$L = p_{uu} \cdot \nu, \qquad M = p_{uv} \cdot \nu, \qquad N = p_{vv} \cdot \nu.$$

Proposition 3.10 (The Gauss formula)**.** *Under the situation above, it holds that*

$$\begin{aligned} p_{uu} &= \sigma_u p_u - \sigma_v p_v + L\nu, \\ p_{uv} &= \sigma_v p_u + \sigma_u p_v + M\nu, \\ p_{vv} &= -\sigma_u p_u + \sigma_v p_v + N\nu. \end{aligned}$$

Proof. Since $\{p_u, p_v, \nu\}$ is a basis of $\mathbb{R}^3$ for each $(u, v) \in U$, one can write

$$p_{uu} = ap_u + bp_v + c\nu, \tag{3.10}$$

where a, b, c are smooth functions on U. Here, since ν is a unit vector perpendicular to both p_u and p_v, we have

$$c = p_{uu} \cdot \nu = L.$$

On the other hand, by (3.7), we have

$$\begin{aligned} e^{2\sigma}a &= p_{uu} \cdot p_u = \frac{1}{2}(p_u \cdot p_u)_u = \frac{1}{2}(e^{2\sigma})_u = \sigma_u e^{2\sigma}, \\ e^{2\sigma}b &= p_{uu} \cdot p_v = (p_u \cdot p_v)_u - p_u \cdot p_{uv} = -\tfrac{1}{2}(p_u \cdot p_u)_v = -\sigma_v e^{2\sigma}. \end{aligned}$$

Thus the first equality of the conclusion is obtained. The second and third equality can be obtained in the same manner. □

Proposition 3.11 (The Weingarten formula)**.** *Under the situation above, it holds that*

$$\nu_u = -e^{-2\sigma}(Lp_u + Mp_v), \qquad \nu_v = -e^{-2\sigma}(Mp_u + Np_v).$$

Proof. If we write $\nu_u = ap_u + bp_v + c\nu$, we have

$$\begin{aligned} e^{2\sigma}a &= \nu_u \cdot p_u = (\nu \cdot p_u)_u - \nu \cdot p_{uu} = -L, \\ e^{2\sigma}b &= \nu_u \cdot p_v = (\nu \cdot p_v)_u - \nu \cdot p_{uv} = -M, \\ c &= \nu_u \cdot \nu = \frac{1}{2}(\nu \cdot \nu)_u, \end{aligned}$$

and the first equality of the conclusion is obtained. The second equality can be proven in the same manner. □

Gauss Frame. As seen in the proofs of Proposition 3.10 and 3.11, $\{p_u, p_v, \nu\}$ is a basis of $\mathbb{R}^3$ for each $(u,v) \in U$. Regarding p_u, p_v and ν as column vectors, we then have a matrix-valued function

$$\mathcal{F} := (p_u, p_v, \nu) \colon U \longmapsto \mathrm{GL}(3,\mathbb{R}) \subset \mathrm{M}_3(\mathbb{R}). \tag{3.11}$$

We call such an $\mathcal{F}$ the *Gauss frame* of the surface. The following theorem is an immediate consequence of Propositions 3.10 and 3.11:

Theorem 3.12. *Let $p\colon U \to \mathbb{R}^3$ be a regular surface defined on a domain U in the uv-plane, and denote by ν the unit normal vector field of it. Assume that (u,v) is an isothermal coordinate system, and the first and second fundamental forms are written as*

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \quad II = L\,du^2 + 2M\,du\,dv + N\,dv^2. \tag{3.12}$$

Then the Gauss frame $\mathcal{F} := (p_u, p_v, \nu)$ satisfies the following system of linear partial differential equations:

$$\frac{\partial \mathcal{F}}{\partial u} = \mathcal{F}\Omega, \qquad \frac{\partial \mathcal{F}}{\partial v} = \mathcal{F}\Lambda, \tag{3.13}$$
$$\Omega := \begin{pmatrix} \sigma_u & \sigma_v & -e^{-2\sigma}L \\ -\sigma_v & \sigma_u & -e^{-2\sigma}M \\ L & M & 0 \end{pmatrix},$$
$$\Lambda := \begin{pmatrix} \sigma_v & -\sigma_u & -e^{-2\sigma}M \\ \sigma_u & \sigma_v & -e^{-2\sigma}N \\ M & N & 0 \end{pmatrix},$$

Gauss-Codazzi equations. The coefficients Ω and Λ in (3.13) must satisfy the integrability condition (2.2) in Lemma 2.2.

Lemma 3.13. *The matrices Ω and Λ in* (3.13) *satisfy*

$$\Omega_v - \Lambda_u - \Omega\Lambda + \Lambda\Omega = O$$

if and only if

$$\sigma_{uu} + \sigma_{vv} + e^{-2\sigma}(LN - M^2) = 0 \tag{3.14}$$

and

$$L_v - M_u = \sigma_v(L+N) \qquad \text{and} \qquad N_u - M_v = \sigma_u(L+N). \tag{3.15}$$

Proof. A direct computation. □

Thus we have

Theorem 3.14 (The Gauss and Codazzi equatoins)**.** *Let $p\colon U \to \mathbb{R}^3$ be a regular surface defined on a domain U in the uv-plane, and denote by ν the unit normal vector field of it. Assume that (u,v) is an isothermal coordinate system, and the first and second fundamental forms are written as* (3.12)*. Then* (3.14) *and* (3.15) *hold.*

Remark 3.15. The equations (3.14) and (3.15) are called the *Gauss equation* and the *Codazzi equations*, respectively. The Gauss equation is often referred as *Gauss' Theorema Egregium.*

Fundamental Theorem for Surfaces. The following is the special case of the fundamental theorem for surfaces (Theorem 2.13):

Theorem 3.16. *Let $U \subset \mathbb{R}^2$ be a simply connected domain, and let σ, L, M, N be C^∞-functions satisfying* (3.14) *and* (3.15). *Then there exists a parametrization $p\colon U \to \mathbb{R}^3$ of regular surface whose fundamental forms are given by* (3.12). *Moreover, such a surface is unique up to orientation preserving isometries of $\mathbb{R}^3$.*

Proof. By Lemma 3.13, Theorem 2.3 yields that there exists a matrix-valued function $\mathcal{F}\colon U \to \mathrm{M}_3(\mathbb{R})$ satisfying (3.13) with the initial condition

$$\mathcal{F}(u_0,v_0) = \begin{pmatrix} e^{\sigma(u_0,v_0)} & 0 & 0 \\ 0 & e^{\sigma(u_0,v_0)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.16}$$

for a fixed point $(u_0,v_0) \in U$. Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ be vector-valued functions such that $\mathcal{F} = (\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$. Since

$$\boldsymbol{a}_v = \sigma_v \boldsymbol{a} + \sigma_u \boldsymbol{b} + M\boldsymbol{c} = \boldsymbol{b}_u,$$

the vector-valued 1-form $\boldsymbol{\omega} := \boldsymbol{a}\,du + \boldsymbol{b}\,dv$ is closed. Then by Poincaré's lemma (Theorem 2.6), there exists a vector-valued function $p\colon U \to \mathbb{R}^3$ such that $dp = \boldsymbol{\omega}$:

$$p_u = \boldsymbol{a}, \qquad p_v = \boldsymbol{b}.$$

Let

$$\hat{\mathcal{F}} := (e^{-\sigma}\boldsymbol{a}, e^{-\sigma}\boldsymbol{b}, \boldsymbol{c}).$$

Then it holds that

$$\hat{\mathcal{F}}_u = \hat{\mathcal{F}}\hat{\Omega}, \qquad \hat{\mathcal{F}}_v = \hat{\mathcal{F}}\hat{\Lambda},$$

$$\hat{\Omega} := \begin{pmatrix} 0 & \sigma_v & -e^{-\sigma}L \\ -\sigma_v & 0 & -e^{-\sigma}M \\ e^{-\sigma}L & e^{-\sigma}M & 0 \end{pmatrix},$$

$$\hat{\Lambda} := \begin{pmatrix} 0 & -\sigma_u & -e^{-\sigma}M \\ \sigma_u & 0 & -e^{-\sigma}N \\ e^{-\sigma}M & e^{-\sigma}N & 0 \end{pmatrix}$$

with $\hat{\mathcal{F}}(u_0,v_0) = \mathrm{id}$. Then by Theorem 2.3, $\hat{\mathcal{F}} \in \mathrm{SO}(3)$ for all $(u,v) \in U$. This means that

$$p_u \cdot p_u = \boldsymbol{a}\cdot\boldsymbol{a} = e^{2\sigma}, \qquad p_u \cdot p_v = \boldsymbol{a}\cdot\boldsymbol{b} = 0, \qquad p_v \cdot p_v = \boldsymbol{b}\cdot\boldsymbol{b} = e^{2\sigma}$$
$$p_u \cdot \nu = p_v \cdot \nu = 0, \qquad \nu\cdot\nu = 1,$$

where $\nu := \boldsymbol{c}$. Hence the first fundamental form of p is $ds^2 = e^{2\sigma}(du^2+dv^2)$ and ν is the unit normal vector field of p. Moreover, since

$$p_{uu}\cdot\nu = a_u \cdot \boldsymbol{c} = L, \qquad p_{uv}\cdot\nu = M, p_{vv}\cdot\nu = N.$$

Thus, p is the desired immersion.

Next, we prove the uniqueness. Let $\tilde{p}$ be an immersion with (3.12). Then the Gauss frame $\widetilde{\mathcal{F}}$ satisfies the equation (3.13) as well as $\mathcal{F}$. Here, $|\tilde{p}_u(u_0,v_0)| = e^{\sigma(u_0,v_0)}$, $|\tilde{p}_v(u_0,v_0)| = e^{\sigma(u_0,v_0)}$, and $\tilde{p}_u$, $\tilde{p}_v$, $\tilde{\nu}$ are mutually perpendicular. Thus, by a suitable rotation in $\mathbb{R}^3$, we may assume $\widetilde{\mathcal{F}}(u_0,v_0)$ coincides with $\mathcal{F}(u_0,v_0)$ without loss of generality. Then $\widetilde{F} = \mathcal{F}$ by the uniqueness part of Theorem 2.3, and $dp = d\widetilde{p}$ holds. Hence $\widetilde{p} = p$ up to additive constant vector. $\square$

Exercises

3-1$^{\mathrm{H}}$ Prove Theorem 3.14.

3-2$^{\mathrm{H}}$ Let $(x(u), z(u))$ be a curve on the xz-plane parametrized by the arc-length parameter (that is, $(\dot{x})^2 + (\dot{z})^2 = 1$). Find an isothermal parameter of the surface of revolution

$$p(u, v) = \bigl(x(u)\cos v, x(u)\sin v, z(u)\bigr).$$