The Hopf Differential

Complexification of vector spaces. Let V be an n-dimensional real vector space. By extending the coefficients to complex numbers, we obtain an n-dimensional complex vector space $V^{\mathbb{C}}$, called the *complexification of* V. More precisely, take a basis $\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\}$ of V. Then $V^{\mathbb{C}}$ is the complex vector space generated by $\{\boldsymbol{a}_j\}$:

$$
\begin{aligned}
V^{\mathbb{C}} &= \{x_1\boldsymbol{a}_1 + \dots + x_n\boldsymbol{a}_n \mid x_j \in \mathbb{C} \quad (j = 1, \dots, n)\} \\
&= \operatorname{Span}_{\mathbb{C}}\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\}.
\end{aligned} \tag{4.1}
$$

This expression does not depend on the choice of $\{\boldsymbol{a}_j\}$. In fact, let $\{\boldsymbol{b}_1, \dots, \boldsymbol{b}_n\}$ be another basis of V and $A \in \mathrm{GL}(n, \mathbb{R})$ the change of bases $\{\boldsymbol{a}_j\}$ and $\{\boldsymbol{b}_j\}$:

$$
(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n) = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_n)A.
$$

Since

$$
\begin{aligned}
\boldsymbol{x} :&= x_1\boldsymbol{a}_1 + \dots + x_n\boldsymbol{a}_n = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\
&= (\boldsymbol{b}_1, \dots, \boldsymbol{b}_n)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \qquad \left(\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} := A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right),
\end{aligned}
$$

we have that $\operatorname{Span}_{\mathbb{C}}\{\boldsymbol{b}_j\} = \operatorname{Span}_{\mathbb{C}}\{\boldsymbol{a}_j\}$.

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The *dual vector space* W^* of a real (complex) vector space W is the set of linear functions on W:

$$
W^* := \{\sigma\colon W \to \mathbb{R} \mid \mathbb{R}\text{-linear}\} \quad \bigl(\text{resp.}\{\sigma\colon W \to \mathbb{C} \mid \mathbb{C}\text{-linear}\}\bigr).
$$

It is easy to see that $(W^{\mathbb{C}})^* = (W^*)^{\mathbb{C}}$.

The complexification $V^{\mathbb{C}}$ is also interpreted as a $2n$-dimensional real vector space spanned by

$$
\boldsymbol{a}_1, \dots, \boldsymbol{a}_n; \quad \mathrm{i}\boldsymbol{a}_1, \dots, \mathrm{i}\boldsymbol{a}_n,
$$

where $\mathrm{i} = \sqrt{-1}$. Under such a situation, V is an n-dimensional subspace of $V^{\mathbb{C}}$ as a real vector space.

Example 4.1. The complexification of $\mathbb{R}^n$ is $\mathbb{C}^n$. In fact, $\mathbb{C}^n = \operatorname{Span}_{\mathbb{C}}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$, where $\{\boldsymbol{e}_j\}$ is the canonical basis of $\mathbb{R}^n$.

2-dimensional case. We assume that V is a real vector space of dimension 2, and take a basis $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$. Then the *dual basis* $\{\alpha_1, \alpha_2\}$ of V^* is defined by

$$
\alpha_j(\boldsymbol{a}_k) = \delta_{jk} = \begin{cases} 1 & (j = k), \\ 0 & (j \neq k) \end{cases},
$$

and

$$
(V^*)^{\mathbb{C}} = \operatorname{Span}_{\mathbb{C}}(\alpha_1, \alpha_2) = \operatorname{Span}_{\mathbb{C}}(\beta, \bar{\beta}),
$$

where

$$
\beta := \alpha_1 + \mathrm{i}\alpha_2, \qquad \bar{\beta} := \alpha_1 - \mathrm{i}\alpha_2.
$$

We set

$$
\boldsymbol{b} := \frac{1}{2}(\boldsymbol{a}_1 - \mathrm{i}\boldsymbol{a}_2), \qquad \bar{\boldsymbol{b}} := \frac{1}{2}(\boldsymbol{a}_1 + \mathrm{i}\boldsymbol{a}_2).
$$

Then $\{\boldsymbol{b}, \bar{\boldsymbol{b}}\}$ is a basis of $V^{\mathbb{C}}$ whose dual basis is $\{\beta, \bar{\beta}\}$.

Then a real vector $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 \in V$ is identified with

$$\xi\boldsymbol{b} + \bar{\xi}\bar{\boldsymbol{b}} = 2\,\mathrm{Re}(\xi\boldsymbol{b}),$$

where $\xi := x_1 + \mathrm{i}x_2$ and $\bar{\xi}$ is its complex conjugate.

Compexified tangent spaces of Riemann surfaces. Let S be a *Riemann surface*, that is, a complex 1-manifold, and take a local complex coordinate neighborhood $(U; z)$ around $p \in S$. Then (u, v) $(z = u + \mathrm{i}v)$ is a real coordinate system on $U \subset S$.

The tangent space T_xS is a real vector space spanned by $\{(\partial/\partial u)_x, (\partial/\partial v)_x\}$, and $\{(du)_x, (dv)_x\}$ is the dual basis of it. Then, as seen in the previous paragraph, the complexification of $(T_xS)^{\mathbb{C}}$ and its dual $(T_x^*S)^{\mathbb{C}}$is obtained as

$$(4.2)\qquad (T_xS)^{\mathbb{C}} = \mathrm{Span}_{\mathbb{C}}\left\{\left(\frac{\partial}{\partial z}\right)_x, \left(\frac{\partial}{\partial \bar{z}}\right)_x\right\}$$

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial u} - \mathrm{i}\frac{\partial}{\partial v}\right), \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial u} + \mathrm{i}\frac{\partial}{\partial v}\right),$$

$$(4.3)\qquad (T_x^*S)^{\mathbb{C}} = \mathrm{Span}_{\mathbb{C}}\{(dz)_x, (d\bar{z})_x\}$$

$$dz := du + \mathrm{i}dv, \quad d\bar{z} := du - \mathrm{i}dv.$$

In particular $\{(dz)_x, (d\bar{z})_x\}$ is the dual basis of $\{(\partial/\partial z)_x, (\partial/\partial\bar{z})_x\}$.

Lemma 4.2. *Let $(U; z = u + \mathrm{i}v)$ be a complex coordinate neighborhood of a Riemann surface S. Then a function $f\colon U \to \mathbb{C}$ is holomorphic if and only if*

$$\frac{\partial f}{\partial \bar{z}}\left(= \frac{1}{2}\left(\frac{\partial f}{\partial u} - \mathrm{i}\frac{\partial f}{\partial v}\right)\right) = 0.$$

Proof. We write $f(u, v) = \xi(u, v) + \mathrm{i}\eta(u, v)$, where ξ and η are real-valued function on U. Then

$$2\frac{\partial f}{\partial \bar{z}} = \frac{\partial(\xi + \mathrm{i}\eta)}{\partial u} - \mathrm{i}\frac{\partial(\xi + \mathrm{i}\eta)}{\partial v} = \left(\frac{\partial \xi}{\partial u} - \frac{\partial \eta}{\partial v}\right) + \mathrm{i}\left(\frac{\partial \eta}{\partial u} + \frac{\partial \xi}{\partial v}\right),$$

which vanishes if and only if the map $(u, v) \mapsto (\xi, \eta)$ satisfies the Cauchy-Riemann equation. □

Definition 4.3.

$$(T_xS)^{(1,0)} := \mathrm{Span}_{\mathbb{C}}\{(dz)_x\} \subset (T_x^*S)^{\mathbb{C}},$$
$$(T_xS)^{(0,1)} := \mathrm{Span}_{\mathbb{C}}\{(d\bar{z})_x\} \subset (T_x^*S)^{\mathbb{C}}.$$

Lemma 4.4. $(T_x^*S)^{\mathbb{C}} = (T_x^*S)^{(1,0)} \oplus (T_x^*S)^{(0,1)}$. *Moreover such a decomposition does not depend on a choice of complex coordinate systems.*

Proof. Since $(dz)_x$ and $(d\bar{z})_x$ span $(T_x^*(S))^{\mathbb{C}}$, the first part is obtained. Let w be another complex coordinate. Then one can easily show that

$$dw = \frac{\partial w}{\partial z}dz + \frac{\partial w}{\partial \bar{z}}d\bar{z}, \quad d\bar{w} = \frac{\partial \bar{w}}{\partial z}dz + \frac{\partial \bar{w}}{\partial \bar{z}}d\bar{z}.$$

Since the coordinate change $z \mapsto w$ is holomorphic, Lemma 4.2 yields that

$$\frac{\partial w}{\partial \bar{z}} = 0, \qquad \frac{\partial \bar{w}}{\partial z} = \overline{\frac{\partial w}{\partial \bar{z}}} = 0.$$

Hence, by definition of complex derivation,

$$dw = \frac{dw}{dz}\,dz, \qquad d\bar{w} = \overline{\frac{dw}{dz}}\,d\bar{z}$$

hold. Then the second part of the conclusion follows. □

Symmetric 2-differentials on Riemann surfaces. A *symmetric 2-form* on a real vector space V is a bilinear form

$$\sigma\colon V \times V \longrightarrow \mathbb{R}$$

such that $\sigma(\boldsymbol{x},\boldsymbol{y}) = \sigma(\boldsymbol{y},\boldsymbol{x})$ holds for all $\boldsymbol{x}$, $\boldsymbol{y} \in V$. A *symmetric 2-tensor* or a *symmetric 2-differential* on a smooth manifold S is a correspondence

$$\sigma\colon S \ni x \longmapsto \text{a symmetric 2-form } \sigma_x \text{ on } T_xS$$

such that $\sigma(X,Y)\colon S \to \mathbb{R}$ is smooth for each smooth vector fields X and Y on S. Taking a local coordinate system (u,v) around p, a symmetric 2-tensor σ is expressed as

$$\sigma = s_{11}\,du^2 + 2s_{12}\,du\,dv + s_{22}\,dv^2 \qquad (4.4)$$
$$\begin{pmatrix} s_{11} := \sigma\left(\partial/\partial u, \partial/\partial u\right), & s_{22} := \sigma\left(\partial/\partial v, \partial/\partial v\right), \\ s_{12} = s_{21} := \sigma\left(\partial/\partial u, \partial/\partial v\right) & \end{pmatrix}.$$

Example 4.5 (Surfaces in the Euclidean space)**.** Let $p\colon S \to \mathbb{R}^3$ be an immersion of a Riemann surface S into $\mathbb{R}^3$. Since S is orientable,[9] there exists a (globally defined) unit normal vector field ν which is considered as a map $\nu\colon S \to S^2 \subset \mathbb{R}^3$, called the *Gauss map*.

The *first fundamental form* ds^2 and the *second fundamental form* $I\!I$ are defined as

$$ds^2(\boldsymbol{v},\boldsymbol{w}) := dp(\boldsymbol{v}) \cdot dp(\boldsymbol{w}) \text{ and } I\!I(\boldsymbol{v},\boldsymbol{w}) := -dp(\boldsymbol{v}) \cdot d\nu(\boldsymbol{w}),$$

respectively, for $\boldsymbol{v}$, $\boldsymbol{w} \in T_xS$ $(x \in S)$. Then both ds^2 and $I\!I$ are symmetric 2-differentials on S.

Since $dp(\partial/\partial u) = p_u, \dots,$ and

$$p_u \cdot \nu_u = (p_u \cdot \nu)_u - p_{uu} \cdot \nu,$$
$$p_u \cdot \nu_v = p_v \cdot \nu_u = -p_{uv} \cdot \nu, \quad p_v \cdot \nu_v = -p_{vv} \cdot \nu,$$

the definitions of the fundamental forms here coincide with those as (2.11) in Section 2.

Let $(U; z = u + \mathrm{i}v)$ be a complex chart of a Riemann surface S. By virtue of (4.3), one can rewrite (4.4) as

$$\sigma = \tilde{s}_{20}\,dz^2 + 2\tilde{s}_{11}\,dz\,d\bar{z} + \tilde{s}_{02}d\bar{z}^2, \qquad (4.5)$$

where[10]

$$\tilde{s}_{20} = \frac{s_{11} - s_{22} - 2\mathrm{i}s_{12}}{4},$$
$$\tilde{s}_{02} = \frac{s_{11} - s_{22} + 2\mathrm{i}s_{12}}{4}, \qquad \tilde{s}_{11} = \frac{s_{11} + s_{22}}{4}.$$

[9] A Riemann surface (more generally, a complex manifold) is necessarily orientable. In fact, a holomorphic coordinate change $z = u + \mathrm{i}v \mapsto w = \xi + \mathrm{i}\eta$ has positive Jacobian because of the Cauchy-Riemann equation.

[10] Although the form (4.5) might be written as $\sigma^{\mathbb{C}}$ because it is a complexification of the original σ, we do not distinguish them in this notebook.

Definition 4.6. Let σ be a symmetric 2-differential as in (4.5). Then we set

$$\sigma^{(2,0)} := \tilde{\sigma}_{20} dz^2,\ \sigma^{(1,1)} := 2\tilde{\sigma}_{11} dz\, d\bar{z},\ \sigma^{(0,2)} := 2\tilde{\sigma}_{02} d\bar{z}^2,$$

and call them the $(2,0)$-part, $(1,1)$-part, and $(0,2)$-part of σ, respectively.

Similar to Lemma 4.4,

Lemma 4.7. *The $(2,0)$-part, $(1,1)$-part and $(0,2)$-part of symmetric 2-differnetials are independent on choice of complex coordinates.*

Hopf differentials.

Definition 4.8. An immersion $p\colon S \to \mathbb{R}^3$ is said to be *conformal* if each complex coordinate $z = u + \mathrm{i}v$ corresponds to isothermal coordinate system (u, v).

In the situation of Definition 4.8, the first fundamental form ds^2 is written as

$$ds^2 = e^{2\sigma}(du^2 + dv^2) = e^{2\sigma}\, dz\, d\bar{z}. \tag{4.6}$$

Thus we have

Lemma 4.9. *An immersion $p\colon S \to \mathbb{R}^3$ of a Riemann surface S is conformal if and only if the first fundamental form has no both $(2,0)$-part and $(0,2)$-part.*

Definition 4.10. Let $p\colon S \to \mathbb{R}^3$ be a conformal immersion of a Riemann surface of S. The $(2,0)$-part Q of the second fundamental form is called the *Hopf differential.*

Lemma 4.11. *If the first and second fundamental forms are in the form*

$$\begin{aligned} ds^2 &= e^{2\sigma}(du^2 + dv^2) = e^{2\sigma}\, dz\, d\bar{z}, \\ I\!I &= L\, du^2 + 2M\, du\, dv + N\, dv^2 \end{aligned} \tag{4.7}$$

in the complex coordinate $z = u + \mathrm{i}v$, the Hopf differential Q and the mean curvature H are expressed as

$$Q = \frac{1}{4}\big((L-N) - 2\mathrm{i}M\big)\, dz^2, \qquad H = \frac{e^{-2\sigma}}{2}(L+N). \tag{4.8}$$

Proof. The equation **??** yields the expression of the Hopf differential. Since the representation matrix of the first fundamental form is $e^{2\sigma}\,\mathrm{id}$, then the coefficients of the Weingarten matrix (cf. (**??**) in Section 2) are $e^{-2\sigma}$ times of L, M and N. Since the $2H$ is the trace of the Weingarten matrix, the expression of the mean curvature holds. □

Definition 4.12. Let $p\colon S \to \mathbb{R}^3$ be an immersion of a 2-manifold S. A point $x \in S$ is called an *umbilic point* if the first fundamental form ds^2 and the second fundamental form $I\!I$ are proportional at the point p. If all points of S are umbilic points, p is called *totally umbilic.*

Proposition 4.13 (cf. §7 in [3-1])**.** *The image of a totally umbilic immersion is a part of a plane or a round sphere.*

Proof. Since the first and second fundamental forms are proportional, the Weingarten matrix (**??**) is a scalar multiplication of id: $A = \lambda\,\mathrm{id}$ on a coordinate neighborhood (u, v). Then the derivatives of the unit normal vector field satisfy

$$\nu_u = -\lambda p_u, \qquad \nu_v = -\lambda p_v.$$

Differentiating these, we have

$$\begin{aligned}\nu_{uv} &= -\lambda_v p_u + \lambda p_{uv},\\ \nu_{vu} &= -\lambda_u p_v + \lambda p_{vu}.\end{aligned}$$

This implies $d\lambda = 0$ on a coordinate neighborhood, and thus λ must be constant. When $\lambda = 0$, ν is constant vector, and then the image of p is a part of the plane. If $\lambda \neq 0$, $p + \nu/\lambda$ is constant. This means that the image lies on a sphere of radius $1/|\lambda|$. □

The Gauss and Codazzi equations.

Theorem 4.14. *Let $p\colon S \to \mathbb{R}^3$ be a conformal immersion of a Riemann surface S, and let ds^2, H and Q be the first fundamental form, the mean curvature and the Hopf differential, respectively. Take a complex coordinate $z = u + \mathrm{i}v$ of S, and write*

$$ds^2 = e^{2\sigma}\,dz\,d\bar{z}, \quad Q = q\,dz^2.$$

Then the Gauss equation (3.14) *and the Codazzi equations* (3.15) *are equivalent to*

$$\text{(4.9)} \qquad \frac{\partial^2\sigma}{\partial z\partial\bar{z}} + e^{-2\sigma}q\bar{q} + \frac{1}{4}e^{2\sigma}H^2 = 0, \qquad \frac{\partial q}{\partial\bar{z}} = \frac{e^{2\sigma}}{4}\frac{\partial H}{\partial z},$$

respectively.

Proof. By (4.8),

$$\begin{aligned} q\bar{q} &= \frac{1}{16}\bigl((L-N)^2 + 4M^2\bigr) = \frac{1}{16}\left((L+N)^2 - 4(LN - M^2)\right)\\ &= \frac{1}{4}\left(e^{4\sigma}H^2 - (LN - M^2)\right).\end{aligned}$$

Since

$$\frac{\partial^2}{\partial z\partial\bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right),$$

the Gauss equation (3.14) is equivalent to the first equation of (4.9). The second equation follows from (3.15). □

Corollary 4.15. *Let $p\colon S \to \mathbb{R}^3$ be a conformal immersion of a Riemann surface S with constant mean curvature. Then the Hopf differential $Q = q\,dz^2$ is holomorphic, that is, q is a holomorphic function in z, where z is an arbitrary complex coordinate on S.*

Proof. When $dH = 0$, the second equation of (4.9) implies $q_{\bar{z}} = 0$. □

Since zeros of holomorhpic function are isolated unless the function is identically zero, we have

Corollary 4.16. *An umbilic point of a constant mean curvature surface is isolated unless it is totally umbilic.*

References

[4-1] 梅原雅顕，山田光太郎，曲線と曲面（改訂版），裳華房，2014.

[4-2] Masaaki Umehara and Kotaro Yamada, Differential Geometry of Curves and Surfaces, (trasl. by Wayne Rossman), World Scientific, 2017.

Exercises

4-1$^{\mathrm{H}}$ Let S be a Riemann surface, and let

$$p\colon S \longrightarrow \mathbb{R}^3$$

be a conformal immersion of constant mean curvature without umbilic points. Then for each $x \in D$, there exists a complex coordinate z such that

$$ds^2 = e^{2\sigma}\,dz\,d\bar{z}, \qquad Q = dz^2.$$