Advanced Topics in Geometry E (MTH.B501)

Linear Ordinary Differential Equations

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Ordinary Differential Equations 常微分方程式

unknown

vector valued.

normal form 正規形

(*) $\boxed{\text{solution}}$

$$\frac{d}{dt}x(t) = f\big(t, x(t)\big), \qquad \boxed{x(t_0) = x_0} \text{ initial condition}$$

α (parameter)

$f : I \times U \longrightarrow \mathbb{R}^m$, $\quad I \subset \mathbb{R}$, $\quad U \subset \mathbb{R}^m$

C^∞

1. ► Existence
2. ► Uniqueness
3. ► Regularity on initial conditions and parameters

① $\exists\, x : (t_0 - \varepsilon, t_0 + \varepsilon) \longrightarrow \mathbb{R}^m$ $\quad C^\infty$ $\boxed{\exists \text{ local sol.}}$

② maximal sol. $\quad x : J \overset{\subset I}{\longrightarrow} \mathbb{R}^m$ is unique

③ $x = x(t_0, x_0, \alpha)(t) \; : \; C^\infty$

the initial value problem of an ordinary differential equation

Example

$(m=1)$

$$\circledast \qquad \frac{d}{dt}x(t) = f(t, x(t)) = \lambda x(t), \qquad x(0) = x_0.$$

$\boxed{\text{linear in } x}$ λ: const. $t_0 = 0$

$x(t) = x_0 \cdot \exp(\lambda t)$. is the unique solution of $\circledast$

on $\mathbb{R}$

Example

$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ $n = 2$

$\mathbb{R}$

$$\Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \cos\omega t + \frac{y_0}{\omega}\sin\omega t \\ -x_0\omega \sin\omega t + y_0 \cos\omega t \end{pmatrix}$$

linear in (x, y)

$f(t, x, y) = \begin{pmatrix} y \\ -\omega^2 x \end{pmatrix}$

$$(*) \begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\omega^2 x \end{cases}$$

$x(0) = x_0$

$y(0) = y_0$

$$\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \left(\frac{d^2x}{dt^2} = -\omega^2 x\right)$$

Example

the logistic eq.

1-1

$$\boxed{\frac{dx}{dt} = \lambda x(a-x)}$$

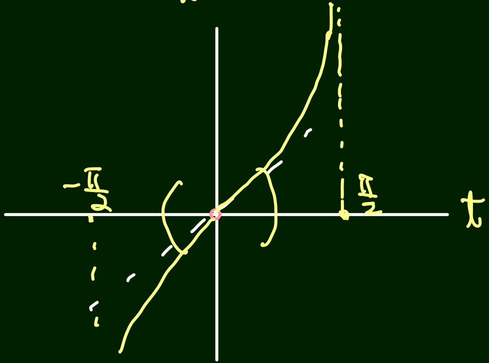
$$\frac{dx}{dt} = 1 + x^2, \qquad x(0) = 0.$$

$$\frac{dx}{dt} = f(t,x) = 1 + x^2$$

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

UH

$$x(t) = \tan t \; ; \; t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$



$$\frac{dx}{dt} = e^t(1+x^2)$$

$$x = \tan e^t$$