Advanced Topics in Geometry E (MTH.B501)

Linear Ordinary Differential Equations

Kotaro Yamada
kotaro@math.titech.ac.jp
http://www.math.titech.ac.jp/~kotaro/class/2022/geom-e/

Tokyo Institute of Technology

2022/04/19

Linear ordinary differential equations

unknown $x : I \to \mathbb{R}^m$

$$\frac{d}{dt}\boldsymbol{x}(t) = A(t)\boldsymbol{x}(t) + \boldsymbol{b}(t),$$

known data

$\mathbb{R}^m$ valued function

$m \times m$ matrix valued function

continuous

$\boxed{C^r \text{ on } I}$

- Global Existence

$x(0) = x_0$: initial condition

$\underline{\exists!\ \text{solution}}$ $x : I \xrightarrow[C^{r+1}]{} \mathbb{R}^m$

Recall : $\frac{dx}{dt} = f(t, x(t))$ Linear

Proof of Existence : $X := \{x : J \to \mathbb{R}^n ; \text{conti}\}$

$\|e^{-kt} \cdot \|$: the uniform norm. complete

$\Phi : X \ni x \longmapsto x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\, d\tau \in X$

Φ : a contraction map if J is $\boxed{\text{sufficiently small}}$

缩小写像

$\Rightarrow \exists$ fixed pt. $x = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\, d\tau$

($\exists k$ s.t. Φ is contraction) unknown

$$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad \underline{X(t_0) = X_0,}$$

initial condition $\cong \mathbb{R}^{n^2}$

X : $M_n(\mathbb{R})$ - valued ; $M_n(\mathbb{R}) = \{ n \times n \text{ matrices}/\mathbb{R} \}$

Given data

Ω : $\mathbb{R} \supset I \xrightarrow[C^\infty]{} M_n(\mathbb{R})$

B :

The case $B = O$: homogenious (同次・斉次)

Preliminaries

Proposition (Prop. 1.8)

Assume two C^∞ matrix-valued functions $X(t)$ and $\Omega(t)$ satisfy

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0.$$

Then

$$\det X(t) = \underbrace{(\det X_0)}_{\neq 0} \exp \overbrace{\int_{t_0}^{t} \operatorname{tr} \Omega(\tau)\, d\tau}^{\neq 0}.$$

In particular, if $X_0 \in \mathrm{GL}(n, \mathbb{R})$, then $X(t) \in \mathrm{GL}(n, \mathbb{R})$ for all t.

$\mathrm{GL}(n,\mathbb{R}) = \{X \in M_n(\mathbb{R})\,;\, \det X \neq 0\}$ = the set of regular matrices

the general linear group

- $$\frac{dX}{dt} = X\Omega \qquad X(t_0) = X_0$$

余因子行列

$\tilde{X}$: the cofactor matrix of X

$$X\tilde{X} = \tilde{X}X = (\det X)\,\mathrm{id}$$

$$\frac{d}{dt}(\det X) = \operatorname{trace}\left(\tilde{X}\frac{dX}{dt}\right)$$

$$= \operatorname{trace} \tilde{X}X\Omega$$

$$= \operatorname{trace} (\det X)\cdot\Omega$$

$$= (\det X)\cdot\operatorname{trace}\Omega$$

$$\det X = (\det X_0)\cdot \exp\int_{t_0}^{t} \operatorname{trace}\Omega(\tau)\,d\tau$$

Preliminaries

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0.$$

Corollary (Cor. 1.9)

If $\operatorname{tr}\Omega(t) = 0$*, then* $\det X(t)$ *is constant. In particular, if* $X_0 \in \boxed{\mathrm{SL}(n,\mathbb{R})}$*,* X *is a function valued in* $\mathrm{SL}(n,\mathbb{R})$*.*

Lie algebra

$\| \quad \{X \in M_n(\mathbb{R})\,;\ \det X = 1\}$

the special linear group.

$\mathrm{Lie}(\mathrm{SL}(n,\mathbb{R}))$

$\cong \{\Omega\,;\ \operatorname{tr}\Omega = 0\}$

Preliminaries

the transposition

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0.$$

Ω is skew symmetric

Proposition (Prop. 1.10)

Assume $\boxed{{}^t\Omega + \Omega = O}$.
If $X_0 \in \mathrm{O}(n)$ (*resp.* $X_0 \in \mathrm{SO}(n)$),
then $X(t) \in \mathrm{O}(n)$ (*resp.* $X(t) \in \mathrm{SO}(n)$) *for all* t.

(det = 1)

$O(n) \cap SL(n, \mathbb{R})$

const. ${}^tX = X^{-1}$

$= \{X \in M_n(\mathbb{R}) \,;\, X{}^tX = {}^tXX = id\}$ → det = ±1

the set of orthogonal matrices

$$\frac{d}{dt}(X{}^tX) = X(\Omega + {}^t\Omega){}^tX = 0$$

Linear ordinary differential equations.

Proposition (Prop. 1.12)

Let $\Omega(t)$ be a C^∞-function valued in $\mathrm{M}_n(\mathbb{R})$ defined on an interval I. Then for each $t_0 \in I$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0,\mathrm{id}}(t)$ such that

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = \mathrm{id}\,.$$

Corollary (Cor. 1.13)

There exists the unique matrix-valued C^∞-function $X_{t_0,X_0}(t)$ defined on I such that

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \quad X(t_0) = X_0 \quad \big(X(t) := X_{t_0,X_0}(t)\big)$$

In particular, $X_{t_0,X_0}(t)$ is of class C^∞ in X_0 and t.

- $\frac{dY}{dt} = Y\Omega \qquad Y(t_0) = \mathrm{id}$
- $\boxed{X = X_0 Y} \qquad \frac{dX}{dt} = X_0 \frac{dY}{dt} = X_0 Y \Omega = X\Omega$

$X(t_0) = X_0 Y(t_0) = X_0\, \mathrm{id} = X_0$

Non-homogenious case

Proposition (Prop. 1.14)

Let $\Omega(t)$ and $B(t)$ be matrix-valued C^∞-functions defined on I. Then for each $t_0 \in I$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function defined on I satisfying

- $$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad X(t_0) = X_0.$$

inhomogenious

Fundamental Theorem

Theorem (Thm. 1.15)

$n \times n$

Let I and U be an interval and a domain in $\mathbb{R}^m$, respectively, and let $\Omega(t, \boldsymbol{\alpha})$ and $B(t, \boldsymbol{\alpha})$ be matrix-valued C^∞-functions defined on $I \times U$ ($\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$). Then for each $t_0 \in I$, $\boldsymbol{\alpha} \in U$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0, X_0, \boldsymbol{\alpha}}(t)$ defined on I such that

$$\frac{dX(t)}{dt} = X(t)\Omega(t, \boldsymbol{\alpha}) + B(t, \boldsymbol{\alpha}), \qquad X(t_0) = X_0. \tag{1}$$

Moreover,

$$I \times I \times \mathrm{M}_n(\mathbb{R}) \times U \ni (t, t_0, X_0, \boldsymbol{\alpha}) \mapsto X_{t_0, X_0, \boldsymbol{\alpha}}(t) \in \mathrm{M}_n(\mathbb{R})$$

is a C^∞-map.

Application to Space Curves (空間曲線の基本定理)

幾何学概論I

1次元

- $\gamma \colon I \to \mathbb{R}^3$: a space curve parametrized by the arclength. 弧長
- $e := \gamma'$ $|e| = 1$
- $\kappa = |e'|$; we assume $\kappa > 0$ (the curvature) 曲率 $|\gamma''|$
- $n = e'/\kappa$ (the principal normal) 主法線
- $b = e \times n$ (the binormal) 従法線
- $\tau = -b' \cdot n$ (the torsion) 捩率

inner product

$$\boxed{(\kappa, \tau) \rightsquigarrow \gamma}$$ Fundamental theorem

Frenet-Serret

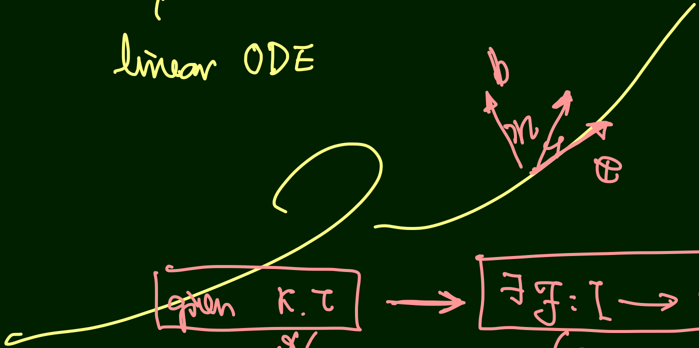
3 × 3 orthogonal matrices with det = 1

- $\mathcal{F} := (\boldsymbol{e}, \boldsymbol{n}, \boldsymbol{b})\colon I \to \mathrm{SO}(3)$: the Frenet Frame 枠

$$\boxed{\frac{d\mathcal{F}}{ds} = \mathcal{F}\Omega}, \qquad \Omega = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$$

skew symmetric

linear ODE



$\boxed{\text{given } \kappa, \tau} \longrightarrow \boxed{\exists\, \mathcal{F}\colon I \longrightarrow SO(3)}$

$(\mathcal{F}(t_0) = id)$

γ'

$\mathcal{F} = (e, n, b)$

The Fundamental Theorem for Space Curves

Theorem (Thm. 1.17)

Let $\kappa(s)$ and $\tau(s)$ be C^∞-fnctions defined on an interval I satisfying $\kappa(s) > 0$ on I.
Then there exists a space curve $\gamma(s)$ parametrized by arc-length whose curvature and torsion are κ and τ, respectively.
Moreover, such a curve is unique up to transformation $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$) of $\mathbb{R}^3$.

$\kappa, \tau \mapsto \mathcal{F} : I \to SO(3)$ ambiguity

Frenet Serret

$= (\mathbf{e}\ \mathbf{n}\ \mathbf{b})$

$\mathcal{F} \mapsto A\mathcal{F}$ $(A \in SO(3))$

$\mapsto \gamma(s) = \int_{s_0}^{s} \mathbf{e}(\sigma)\, d\sigma$ ambiguity

$\gamma \mapsto \gamma + \mathbf{b}$

$$\kappa = \frac{a}{1+s^2} \qquad \tau = \frac{b}{1+s^2}$$

$\downarrow$

curve ?

constant

$$\frac{d\mathfrak{F}}{ds} = \frac{1}{1+s^2}\,\mathfrak{F}\,\Omega_0$$

$$\boxed{\frac{d\mathfrak{F}}{d\tau} = \mathfrak{F}\,\Omega_0}$$