Advanced Topics in Geometry E (MTH.B501)

Integrability Conditions

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Exercise 1-1

Problem (Ex. 1-1)

Find the maximal solution of the initial value problem

positive

$$\frac{dx}{dt} = \lambda x(a - x), \qquad x(0) = b,$$

(non-linear)

where λ and a are positive constants, and b is a real number.

$$x(t) = \frac{ab}{(a-b)e^{-a\lambda t} + b}$$

the logistic equation

$0 < x \ll a$

$\Rightarrow x' = \mu x$

(exponential growth)

$$x(t) = \frac{ab}{(a-b)e^{-a\lambda t} + b}$$

(uniqueness)

$$\lg\left(\frac{x}{a-x}\right) = a\lambda t + \text{const}$$

$$x' = \lambda x(a-x)$$

$$\frac{dx}{x(a-x)} = \lambda dt$$

$$\lg \frac{b}{a-b} = \text{const}$$

$$\|$$

$$\frac{1}{a}\left(\frac{1}{x} + \frac{1}{a-x}\right)dx$$

Defined on $\mathbb{R}$ if $0 \leq b \leq a$

Exercise 1-3

Problem (Ex. 1-3)

Find an explicit expression of a space curve $\gamma(s)$ parametrized by the arc-length s, whose curvature and torsion are $a/(1+s^2)$ and $b/(1+s^2)$, respectively, where a and b are constants.

$$\mathcal{F}^{-1}\frac{d\mathcal{F}}{ds} = \frac{1}{1+s^2}\begin{pmatrix} 0 & -a & 0 \\ a & 0 & -b \\ 0 & b & 0 \end{pmatrix}$$

$$\frac{\cancel{ds}}{du}(1+s^2)\frac{d\mathcal{F}}{\cancel{ds}} = \mathcal{F}\overset{\Omega_0}{\begin{pmatrix} 0 & -a & 0 \\ a & 0 & -b \\ 0 & b & 0 \end{pmatrix}}$$

↑ constant matrix

$$u := \tan^{-1} s \qquad \left(\frac{du}{ds} = \frac{1}{1+s^2}\right)$$

$$\frac{d\mathcal{F}}{du} = \mathcal{F}\,\Omega_0$$

constant coefficients.

$$\mathcal{F} = \exp u\,\Omega_0$$

matrix exponenti

$$= \exp \tan^{-1}s\,\Omega_0 = \begin{pmatrix} \mathbf{t}(s) & \mathbf{n}(s) & \mathbf{b}(s) \end{pmatrix}$$

- $\gamma = \int \mathbf{t}\, ds$