Advanced Topics in Geometry E (MTH.B501)

Integrability Conditions

可積分条件

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Key

$$f_{xy} = f_{yx}$$

compatibility cond.
適合条件.

Integrability Conditions

over determined partial differential eq

$n\times n$ matrix-valued

initial condition

$$\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j=1,\dots,m), \qquad X(\mathrm{P}_0) = X_0. \qquad (*)$$

($m\times n^2$ - equality) on $U \subset \mathbb{R}^m$ (domain)

for n^2 - unknowns

Proposition (Prop. 2.3)

If a matrix-valued C^∞ function $\exists$ $X\colon U \to \mathrm{GL}(n,\mathbb{R})$ satisfies $(*)$, it holds that

solution

regular matrices

$$\boxed{\frac{\partial\Omega_j}{\partial u^k} - \frac{\partial\Omega_k}{\partial u^j} = \Omega_j\Omega_k - \Omega_k\Omega_j} \qquad ⊛$$

for each (j,k) with $1 \leq j < k \leq m$.

(compatibility) integrability condition

$\boxed{\text{Our goal}}$: U : $\boxed{\text{simply connected}}$, ⊛

Thm 2.5 $\Rightarrow$ $\exists^1$ $X : U \to M_n(\mathbb{R})$ satisfying $(*)$

- $$\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \cdots, m)$$

$$X^{-1}\frac{\partial}{\partial u^k}\frac{\partial X}{\partial u^j} = \frac{\partial X}{\partial u^k}\Omega_j + X\frac{\partial \Omega_j}{\partial u^k}$$

$$\| \qquad = \cancel{X^{-1}X}\left(\Omega_k\Omega_j + \frac{\partial \Omega_j}{\partial u^k}\right)$$

$$\shortparallel$$

$$X^{-1}\frac{\partial}{\partial u^j}\frac{\partial X}{\partial u^k} = \cancel{X^{-1}X}\left(\Omega_j\Omega_k + \frac{\partial \Omega_k}{\partial u^j}\right)$$

Lemma (Lem. 2.4)

Let $\Omega_j\colon U \to \mathrm{M}_n(\mathbb{R})$ $(j = 1, \ldots, m)$ *be* C^∞*-maps defined on a domain* $U \subset \mathbb{R}^m$ *which satisfy*

$$\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j \Omega_k - \Omega_k \Omega_j.$$

Then for each smooth map

$$\sigma\colon D \ni (t, w) \longmapsto \sigma(t, w) = (u^1(t, w), \ldots, u^m(t, w)) \in U$$

defined on a domain $D \subset \mathbb{R}^2$*, it holds that*

$$\frac{\partial T}{\partial w} - \frac{\partial W}{\partial t} - TW + WT = 0,$$

where $T := \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial u^j}{\partial t}$, $W := \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial u^j}{\partial w}$, $(\widetilde{\Omega}_j := \Omega_j \circ \sigma)$.

Integrability Conditions

$$\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \dots, m), \qquad X(\mathrm{P}_0) = X_0. \qquad (*)$$

Lemma (Lem. 2.1)

Let $X\colon U \to \mathrm{M}_n(\mathbb{R})$ be a C^∞-map satisfying (*). Then for each smooth path $\gamma\colon I \to U$ defined on an interval $I \subset \mathbb{R}$, $\hat{X} := X \circ \gamma : I \to \mathrm{M}_n(\mathbb{R})$ satisfies the ordinary differential equation

$$\frac{d\hat{X}}{dt}(t) = \hat{X}(t)\Omega_\gamma(t) \quad \left(\Omega_\gamma(t) := \sum_{j=1}^{n} \Omega_j \circ \gamma(t) \frac{du^j}{dt}(t)\right)$$

on I, where $\gamma(t) = \bigl(u^1(t), \dots, u^m(t)\bigr)$.

$P_0 \in U \subset (\mathbb{R}^m ; (u^1, \dots, u^m))$

$\Omega_j : U \to M_n(\mathbb{R})$ C^∞

a domain (arcwise connected open)

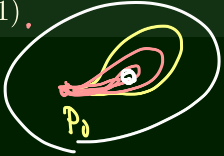
$$\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \dots, m), \qquad X(\mathrm{P}_0) = X_0. \tag{1}$$

$$\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j\Omega_k - \Omega_k\Omega_j \tag{2}$$

Theorem (Thm. 2.5)

Let $\Omega_j : U \to \mathrm{M}_n(\mathbb{R})$ $(j = 1, \dots, m)$ be C^∞-functions defined on a simply connected domain $U \subset \mathbb{R}^m$ satisfying (2). Then for each $\mathrm{P}_0 \in U$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique $n \times n$-matrix valued function $X : U \to \mathrm{M}_n(\mathbb{R})$ satisfying (1).

U



$\forall$ loops are contractible

可縮 (i.e. homotopy equivalent to a point)

Proof of Thm 2.5 (outline)

- Determine $X(Q)$ for $Q \in \mathcal{U}$

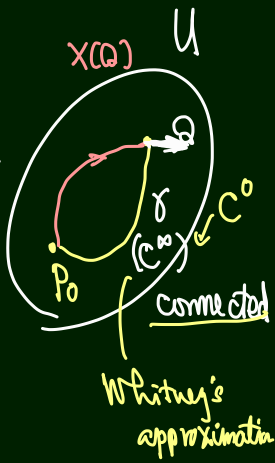
  ① Solve the linear ODE along a path γ joining P_0 & Q

  X_γ: the terminal value

  ② Show that X_γ does not depend on γ.

  - simple connectedness
  - compatibility.

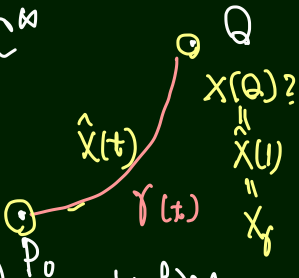
  ③ Show $Q \mapsto X(Q)$: the desired solution.



Exercise: compute $\left(\frac{\partial X}{\partial u^i}\right)$

① $\gamma(t) = (u^1(t), \dots, u^m(t))$; C^∞

$\gamma(0) = P_0$, $\gamma(1) = Q$



• If X satisfies $\dfrac{\partial X}{\partial u^j} = X\,\Omega_j$

Then $\hat{X} = X \circ \gamma = X(u^1(t), \dots, u^m(t))$ satisfies

$\boxed{\text{Lemma 2.1}}$ (*) $\dfrac{d\hat{X}}{dt} = \hat{X}\,\Omega_\gamma$ linear $\qquad \Omega_\gamma := \displaystyle\sum_{j=1}^{m} \Omega_j \circ \gamma \, \frac{du^j}{dt}$

$\dfrac{d\hat{X}}{dt} = \displaystyle\sum_j \frac{\partial X}{\partial u^j} \frac{du^j}{dt}$ chain rule

Solve (*) with initial cond $\hat{X}(0) = X_0$,

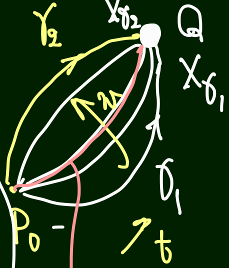
and set $X_\gamma := \hat{X}(1)$

② <u>Claim</u> $X_{\gamma_1} = X_{\gamma_2}$

Whitney $\downarrow$ (U)

- $\sigma : \overset{t}{[0,1]} \times \overset{w}{[0,1]} \longrightarrow U \quad (C^\infty)$

$$\left(\begin{aligned} &\sigma(t,0) = \gamma_1(t) \\ &\sigma(t,1) = \gamma_2(t) \\ &\sigma(0,w) = P_0 \qquad \underline{\sigma(1,w) = Q} \end{aligned}\right)$$



$\sigma(\cdot\ w)$

$w \in [0.1]$ fix

C^∞ homotopy joining γ_1 & γ_2

- $\underline{\tilde{X}(t,w)}$:
$$\left\{\begin{aligned} &\frac{d\tilde{X}}{dt}(\cdot\ \overset{\text{parameter}}{w}) = \tilde{X}\,\Omega_{\sigma(\cdot\, w)} \\ &\tilde{X}(0,w) = X_0 \end{aligned}\right.$$

$$\tilde{X} : \overset{t}{[0.1]} \times \overset{w}{[0.1]} \longrightarrow M_n(\mathbb{R}) \quad \overset{\uparrow}{C^\infty}$$

Claim $\tilde{X}(1,w)$: constant in w.

$T := \Omega_{\sigma(\cdot,w)} = \sum_{j=1}^{m} \Omega_j \frac{\partial u^j}{\partial t}$ $\qquad$ $\sigma(t,w) = (u^1, \dots, u^m)$

$\boxed{\frac{\partial \tilde{X}}{\partial t} = \tilde{X} T}$

• $W := \sum_{j=1}^{m} \Omega_j \frac{\partial u^j}{\partial w}$

$\boxed{\text{We shall prove } \frac{\partial \tilde{X}}{\partial w} = \tilde{X} W}$ ← compatibility

$\Rightarrow$ the Claim follows ∵ $W(1,w) = 0$

$\Uparrow$

$\sigma(1,w) = Q$

Lemma $T_w - W_t = TW - WT$. (Lem 2.4)

$\uparrow$ compatibility condition

$$(\tilde{X}_w)_t = \tilde{X}_{tw} = (\tilde{X}T)_w = \tilde{X}_w T + \tilde{X} \underline{\underline{T_w}}$$

$$= \tilde{X}_w T + \tilde{X}(W_t + TW - WT)$$

$$= \underline{\tilde{X}_w} T + \underline{(\tilde{X}W)_t} - \cancel{\tilde{X}_t W} + \cancel{\tilde{X}TW} - \underline{\tilde{X}WT}$$

$\tilde{X}T$

$$(\underbrace{\tilde{X}_w - \tilde{X}W}_{Y})_t = (\underbrace{\tilde{X}_w - \tilde{X}W}_{Y})T$$

$$\underline{Y_t = YT} \qquad \underline{Y(0,w) = 0}$$

$$Y = \tilde{X}_w - \tilde{X}W = 0. \qquad \square$$

Application: Poincaré's lemma

Theorem (Poincaré's lemma)

If a differential 1*-form*

$$\omega = \sum_{j=1}^{m} \alpha_j(u^1, \dots, u^m)\, du^j$$

defined on a simply connected domain $U \subset \mathbb{R}^m$ *is closed, that is,* $d\omega = 0$ *holds, then there exists a* C^∞*-function* f *on* U *such that* $df = \omega$*. Such a function* f *is unique up to additive constants.*

$\circledast$: Cauchy-Riemann

Theorem

Let $U \subset \mathbb{C} = \mathbb{R}^2$ be a simply connected domain and $\xi(u,v)$ a C^∞-function harmonic on U. Then there exists a C^∞ harmonic function η on U such that $\boxed{\xi(u,v) + \mathrm{i}\eta(u,v)}$ is holomorphic on U.

$$\boxed{\xi_{uu} + \xi_{vv} = 0}$$

$$\exists\eta \quad \Rightarrow \circledast \begin{cases} \eta_u = -\xi_v \\ \eta_v = \xi_u \end{cases}$$

$$\varphi = e^\eta$$

known

$$\longleftrightarrow \begin{cases} \varphi_u = -\xi_v \varphi \\ \varphi_v = \xi_u \varphi \end{cases}$$

(compatibility $\Delta\xi = 0$) $\boxed{(\times)}$

Example

$\xi(u, v) = e^u \cos v$

$\Delta \xi = 0$ in $\mathbb{R}$

$\Rightarrow \eta = e^u \sin v$

$\xi + i\eta = e^u (\cos v + i \sin v)$

$= e^{u+iv}$

Exercise 2-1

Problem

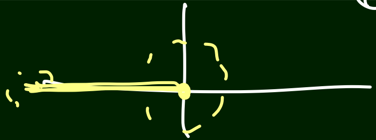
Let $\xi(u,v) := \log\sqrt{u^2+v^2}$ *be a function defined on* $U := \mathbb{R}^2 \setminus \{(0,0)\}$

1. *Show that* ξ *is harmonic on* U.
2. *Find the conjugate harmonic function* η *of* ξ *on*

$$V = \mathbb{R}^2 \setminus \{(u,0) \,|\, u \leqq 0\} \subset U.$$

3. *Show that there exists no conjugate harmonic function of* ξ *defined on* U.

$\mathbb{C}$



Problem

Let $\theta = \theta(u, v)$ be a smooth function on a domain $U \subset \mathbb{R}^2$ such that $0 < \theta < \pi$, and set

$$\Omega := \begin{pmatrix} \theta_u \cot\theta & 0 & \cot\theta \\ -\theta_u \csc\theta & 0 & -\csc\theta \\ 0 & \sin\theta & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} 0 & -\theta_v \csc\theta & -\csc\theta \\ 0 & \theta_v \cot\theta & \cot\theta \\ \sin\theta & 0 & 0 \end{pmatrix}.$$

Prove that the compatibility condition of a system of partial differentail equation

$$\frac{\partial \mathcal{F}}{\partial u} = \mathcal{F}\Omega, \qquad \frac{\partial \mathcal{F}}{\partial v} = \mathcal{F}\Lambda$$

is equivalent to

$$\theta_{uv} = \sin\theta.$$

Exercise 2-3

Problem

Let $\boldsymbol{v} = \boldsymbol{v}(x, y, z)$ *be a vector field defined on a simply connected domain* U *in* $(\mathbb{R}^3; (x, y, z))$*. Assume that* $\boldsymbol{v}$ *is irrotational, that is,* $\operatorname{rot}\boldsymbol{v} = \mathbf{0}$*. Then there exists a function* $\varphi\colon U \to \mathbb{R}$ *such that* $\boldsymbol{v} = \operatorname{grad}\varphi$*.*