Advanced Topics in Geometry E (MTH.B501)

Integrability Conditions

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Integrability Conditions

$$\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \dots, m), \qquad X(\mathrm{P}_0) = X_0. \qquad (*)$$

Proposition (Prop. 2.3)

If a matrix-valued C^∞ function $X\colon U \to \mathrm{GL}(n, \mathbb{R})$ satisfies (*), it holds that

$$\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j\Omega_k - \Omega_k\Omega_j$$

for each (j, k) with $1 \leqq j < k \leqq m$.

Integrability Conditions

Lemma (Lem. 2.4)

Let $\Omega_j \colon U \to \mathrm{M}_n(\mathbb{R})$ $(j = 1, \dots, m)$ *be* C^∞*-maps defined on a domain* $U \subset \mathbb{R}^m$ *which satisfy*

$$\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j \Omega_k - \Omega_k \Omega_j.$$

Then for each smooth map

$$\sigma \colon D \ni (t, w) \longmapsto \sigma(t, w) = (u^1(t, w), \dots, u^m(t, w)) \in U$$

defined on a domain $D \subset \mathbb{R}^2$*, it holds that*

$$\frac{\partial T}{\partial w} - \frac{\partial W}{\partial t} - TW + WT = 0,$$

where $T := \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial u^j}{\partial t}$, $W := \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial u^j}{\partial w}$, $(\widetilde{\Omega}_j := \Omega_j \circ \sigma)$.

Integrability Conditions

$$\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \dots, m), \qquad X(\mathrm{P}_0) = X_0. \qquad (*)$$

Lemma (Lem. 2.1)

Let $X \colon U \to \mathrm{M}_n(\mathbb{R})$ be a C^∞-map satisfying (*). Then for each smooth path $\gamma \colon I \to U$ defined on an interval $I \subset \mathbb{R}$, $\hat{X} := X \circ \gamma : I \to \mathrm{M}_n(\mathbb{R})$ satisfies the ordinary differential equation

$$\frac{d\hat{X}}{dt}(t) = \hat{X}(t)\Omega_\gamma(t) \quad \left(\Omega_\gamma(t) := \sum_{j=1}^{n} \Omega_j \circ \gamma(t) \frac{du^j}{dt}(t)\right)$$

on I, where $\gamma(t) = \bigl(u^1(t), \dots, u^m(t)\bigr)$.

Integrability of Linear systems

$$\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \dots, m), \qquad X(\mathrm{P}_0) = X_0. \tag{1}$$

$$\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j\Omega_k - \Omega_k\Omega_j \tag{2}$$

Theorem (Thm. 2.5)

Let $\Omega_j : U \to \mathrm{M}_n(\mathbb{R})$ $(j = 1, \dots, m)$ *be* C^∞*-functions defined on a simply connected domain* $U \subset \mathbb{R}^m$ *satisfying* (2)*. Then for each* $\mathrm{P}_0 \in U$ *and* $X_0 \in \mathrm{M}_n(\mathbb{R})$*, there exists the unique* $n \times n$*-matrix valued function* $X : U \to \mathrm{M}_n(\mathbb{R})$ *satisfying* (1)

Application: Poincaré's lemma

Theorem (Poincaré's lemma)

If a differential 1*-form*

$$\omega = \sum_{j=1}^{m} \alpha_j(u^1, \dots, u^m)\, du^j$$

defined on a simply connected domain $U \subset \mathbb{R}^m$ *is closed, that is,* $d\omega = 0$ *holds, then there exists a* C^∞*-function* f *on* U *such that* $df = \omega$*. Such a function* f *is unique up to additive constants.*

Application: Conjugation of harmonic functions

Theorem

Let $U \subset \mathbb{C} = \mathbb{R}^2$ be a simply connected domain and $\xi(u,v)$ a C^∞-function harmonic on U. Then there exists a C^∞ harmonic function η on U such that $\xi(u,v) + \mathrm{i}\,\eta(u,v)$ is holomorphic on U.

Application: Conjugation of harmonic functions

Example

$\xi(u, v) = e^u \cos v$

Exercise 2-1

Problem

Let $\xi(u,v) := \log\sqrt{u^2+v^2}$ *be a function defined on* $U := \mathbb{R}^2 \setminus \{(0,0)\}$

1. *Show that* ξ *is harmonic on* U.
2. *Find the conjugate harmonic function* η *of* ξ *on*

$$V = \mathbb{R}^2 \setminus \{(u,0) \,|\, u \leqq 0\} \subset U.$$

3. *Show that there exists no conjugate harmonic function of* ξ *defined on* U.

Exercise 2-2

Problem

Let $\theta = \theta(u, v)$ be a smooth function on a domain $U \subset \mathbb{R}^2$ such that $0 < \theta < \pi$, and set

$$\Omega := \begin{pmatrix} \theta_u \cot\theta & 0 & \cot\theta \\ -\theta_u \csc\theta & 0 & -\csc\theta \\ 0 & \sin\theta & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} 0 & -\theta_v \csc\theta & -\csc\theta \\ 0 & \theta_v \cot\theta & \cot\theta \\ \sin\theta & 0 & 0 \end{pmatrix}.$$

Prove that the compatibility condition of a system of partial differentail equation

$$\frac{\partial \mathcal{F}}{\partial u} = \mathcal{F}\Omega, \qquad \frac{\partial \mathcal{F}}{\partial v} = \mathcal{F}\Lambda$$

is equivalent to

$$\theta_{uv} = \sin\theta.$$

Exercise 2-3

Problem

Let $\boldsymbol{v} = \boldsymbol{v}(x, y, z)$ be a vector field defined on a simply connected domain U in $(\mathbb{R}^3; (x, y, z))$. Assume that $\boldsymbol{v}$ is irrotational, that is, $\operatorname{rot} \boldsymbol{v} = \mathbf{0}$. Then there exists a function $\varphi \colon U \to \mathbb{R}$ such that $\boldsymbol{v} = \operatorname{grad} \varphi$.