# Advanced Topics in Geometry E (MTH.B501)

A review of surface theory

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**Theorem**

*If a differential* $1$*-form*

$\omega = \alpha du + \beta dv$ on $U \subset \mathbb{R}^2$

$$\omega = \sum_{j=1}^{m} \alpha_j(u^1, \dots, u^m)\, du^j$$

单连结

*defined on a simply connected domain* $U \subset \mathbb{R}^m$ *is closed, that is,* $d\omega = 0$ *holds, then there exists a* $C^\infty$*-function* $f$ *on* $U$ *such that* $df = \omega$*. Such a function* $f$ *is unique up to additive constants.*

$du \wedge dv = -dv \wedge du$

$du \wedge du = 0$ closed

$$d\omega = d\alpha \wedge du + d\beta \wedge dv$$
$$= (\alpha_u du + \alpha_v dv) \wedge du + (\beta_u du + \beta_v dv) \wedge dv$$
$$= (\beta_u - \alpha_v)\, du \wedge dv$$

$$d\omega = \underline{(\beta_u - \alpha_v)}\, du \wedge dv = 0$$

$$\Rightarrow \exists f = f(u,v) \quad \text{s.t} \quad df = \omega = \overset{\alpha}{f_u}\, du + \overset{\beta}{f_v}\, dv$$

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$$\circledast \quad \varphi_u = \varphi\alpha \qquad \varphi_v = \varphi\beta \,.$$

1×1

$$\text{integrability} \Leftrightarrow \alpha_v - \beta_u - \cancel{\alpha\beta} + \cancel{\beta\alpha} = 0$$

$$\Leftrightarrow d\omega = 0$$

$$\exists \varphi \text{ satisfying } \circledast \qquad \varphi(u_0, v_0) = 1$$

$$\Rightarrow \varphi \neq 0 \quad \Rightarrow \varphi > 0 \qquad \boxed{(\ \det\varphi \neq 0\ )}$$

$$\Rightarrow f := \log\varphi \qquad f_u = \frac{\varphi_u}{\varphi} = \alpha \quad f_v = \frac{\varphi_v}{\varphi} = \beta.$$

### Theorem

*Let $U \subset \mathbb{C} = \mathbb{R}^2$ be a simply connected domain and $\xi(u, v)$ a $C^\infty$-function harmonic on $U$. Then there exists a $C^\infty$ harmonic function $\eta$ on $U$ such that $\xi(u, v) + \mathrm{i}\,\eta(u, v)$ is holomorphic on $U$.*

正則

$u + iv$

$$\xi = \xi(u, v) \qquad \xi_{uu} + \xi_{vv} = 0$$

Solve $\begin{cases} \eta_u = -\xi_v \\ \eta_v = \xi_u \end{cases}$ (Cauchy-Riemann)

given

$$\Leftrightarrow d\eta = \eta_u\,du + \eta_v\,dv = \underbrace{-\xi_v\,du + \xi_u\,dv}_{\omega}$$

$$d\omega = (\xi_{uu} + \xi_{vv})\,du \wedge dv$$

# Exercise 2-1

## Problem (Ex. 1-1)

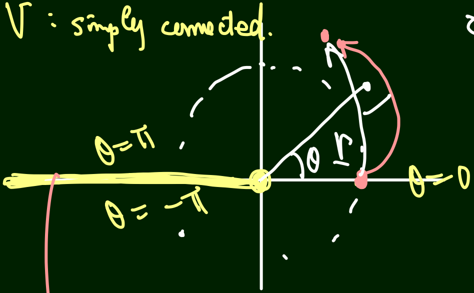
*Let* $\xi(u,v) := \log\sqrt{u^2+v^2}$ *be a function defined on* $U := \mathbb{R}^2 \setminus \{(0,0)\}$

1. *Show that* $\xi$ *is harmonic on* $U$*.*
2. *Find the conjugate harmonic function* $\eta$ *of* $\xi$ *on*

$$V = \mathbb{R}^2 \setminus \{(u,0) \,|\, u \leqq 0\} \subset U.$$

3. *Show that there exists no conjugate harmonic function of* $\xi$ *defined on* $U$*.*

$V$ : simply connected.




$$\xi = \log \sqrt{U^2+V^2}$$
$$= \log r$$

$$\xi_{uu} + \xi_{vv} = 0$$

$$\eta = \theta = \begin{cases} \tan^{-1} \dfrac{V}{U} & (-\pi < \theta < \pi) \\ \cdots ? & \end{cases}$$

$\theta$ cannot be extended continuously.

$$\left( \xi = \frac{U}{U^2+V^2} \Rightarrow \eta = \frac{-V}{U^2+V^2} \right.$$

## Exercise 2-2

$\tan^{-1}\theta$ ?

$$\left( \cot\theta = \frac{1}{\tan\theta} \quad \sec\theta = \frac{1}{\cos\theta} \quad \csc\theta = \operatorname{cosec}\theta = \frac{1}{\sin\theta} \right)$$

### Problem (Ex. 2-2)

*Let $\theta = \theta(u, v)$ be a smooth function on a domain $U \subset \mathbb{R}^2$ such that $0 < \theta < \pi$, and set*

$$\Omega := \begin{pmatrix} \theta_u \cot\theta & 0 & \cot\theta \\ -\theta_u \csc\theta & 0 & -\csc\theta \\ 0 & \sin\theta & 0 \end{pmatrix}, \Lambda := \begin{pmatrix} 0 & -\theta_v \csc\theta & -\csc\theta \\ 0 & \theta_v \cot\theta & \cot\theta \\ \sin\theta & 0 & 0 \end{pmatrix}.$$

*Prove that the compatibility condition of a system of partial differentail equation* 3+3 $\Omega_v - \Lambda_u - \Omega\Lambda + \Lambda\Omega = 0$

$$\frac{\partial \mathcal{F}}{\partial u} = \mathcal{F}\Omega, \qquad \frac{\partial \mathcal{F}}{\partial v} = \mathcal{F}\Lambda$$

$\Uparrow\Downarrow$

*is equivalent to*

$$\boxed{\theta_{uv} = \sin\theta.}$$

sine Gordon eq.

$\theta_{uv} = \lambda\theta$ Klein-Gordon

# Exercise 2-3

## Problem (Ex. 2-3)

*Let* $\boldsymbol{v} = \boldsymbol{v}(x, y, z)$ *be a vector field defined on a simply connected domain* $U$ *in* $(\mathbb{R}^3; (x, y, z))$. *Assume that* $\boldsymbol{v}$ *is irrotational, that is,* $\operatorname{rot} \boldsymbol{v} = \mathbf{0}$. *Then there exists a function* $\varphi\colon U \to \mathbb{R}$ *such that* $\boldsymbol{v} = \operatorname{grad} \varphi$.

$v$ : velocity vector field of steady (irrotational flow)

$$\boxed{\operatorname{rot} v = 0}$$

$$v = \operatorname{grad} \overset{\exists}{\varphi} \leftarrow \text{potential.}$$

$$v = (v_1, v_2, v_3) \Leftrightarrow \omega = v_1\,dx + v_2\,dy + v_3\,dz$$

$$\operatorname{rot} v = 0 \Leftrightarrow d\omega = 0$$

$$v = \operatorname{grad} \varphi \qquad \exists \varphi \text{ s.t. } \omega = d\varphi$$

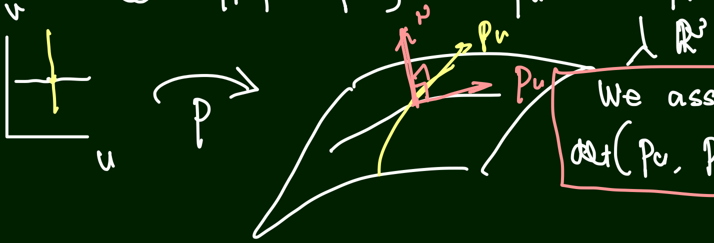
# Immersed surfaces (はめこまれた曲面)

- $p\colon U \to \mathbb{R}^3$: a regular surface 正則曲面 $\Longleftrightarrow$ ①

  $U \subset (\mathbb{R}^2; u, v)$
- $\nu\colon U \to \mathbb{R}^3$: the unit normal vector field. 単位法ベクトル場 ②

① $\{p_u(u,v),\ p_v(u,v)\}$: lin indep. for each $(u,v) \in U$

② $|\nu| = 1$ ; $\nu \cdot p_u = \nu \cdot p_v = 0$




$$\boxed{\text{We assume } \det(p_u,\ p_v,\ \nu) > 0}$$

## Fundammental forms

vector-valued

$$(p_u du + p_v dv) \cdot (p_u du + p_v dv)$$

$$ds^2 := dp \cdot dp = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$ 1st fund form

$E = (p_u \cdot p_u)$, $F = p_u \cdot p_v$, $G = (p_v \cdot p_v)$

$$\begin{pmatrix} p_u\cdot p_u & p_u\cdot p_v \\ p_v\cdot p_u & p_v\cdot p_v \end{pmatrix} = \widehat{I} := \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} {}^t p_u \\ {}^t p_v \end{pmatrix} (p_u, p_v),$$

The inner prod $p_u \cdot p_u = {}^t p_u p_u$

$$II := -d\nu \cdot dp = \cancel{-}\, L\,du^2 + 2M\,du\,dv + N\,dv^2,$$ 2nd fund form

$$\widehat{II} := \begin{pmatrix} L & M \\ M & N \end{pmatrix} = -\begin{pmatrix} {}^t p_u \\ {}^t p_v \end{pmatrix} (\nu_u, \nu_v)$$

$$= -\begin{pmatrix} p_u\cdot\nu_u & p_u\cdot\nu_v \\ p_v\cdot\nu_u & p_v\cdot\nu_v \end{pmatrix}$$

$p_v\cdot\nu_u = p_u\cdot\nu_v$

$$p_u\cdot\nu_v = \underline{(\underbrace{p_u\cdot\nu}_{=0})_v} - p_{uv}\cdot\nu = -p_{vu}\cdot\nu$$

$$\boxed{EG - F^2 = |p_u \times p_v|^2 > 0}$$

Weingarten matrix

$$A := \widehat{I}^{-1}\,\widehat{II} = \begin{pmatrix} A^1_1 & A^1_2 \\ A^2_1 & A^2_2 \end{pmatrix}$$

~~$\lambda_1, \lambda_2$ : the eigenvalues of $A$~~

$\lambda_1, \lambda_2$ : eigenval.

$$K := \lambda_1\lambda_2 = \det A = \frac{\det \widehat{II}}{\det \widehat{I}}$$

Gaussian curvature

$$H := \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{2}\operatorname{tr} A.$$

mean curvature.

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The fundamental theorem for surface

" $\hat{I}$ $\hat{II}$ determine the surface "

Gauss-Weingarten eq

$\boxed{\text{Fundamental equation}}$

$\mathcal{F}_u = \mathcal{F}\Omega,\ \mathcal{F}_v = \mathcal{F}\Lambda$