Advanced Topics in Geometry E (MTH.B501)

A review of surface theory

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Application: Poincaré's lemma

Theorem

If a differential 1*-form*

$$\omega = \sum_{j=1}^{m} \alpha_j(u^1, \dots, u^m)\, du^j$$

defined on a simply connected domain $U \subset \mathbb{R}^m$ *is closed, that is,* $d\omega = 0$ *holds, then there exists a* C^∞*-function* f *on* U *such that* $df = \omega$*. Such a function* f *is unique up to additive constants.*

Application: Conjugation of harmonic functions

Theorem

Let $U \subset \mathbb{C} = \mathbb{R}^2$ be a simply connected domain and $\xi(u, v)$ a C^∞-function harmonic on U. Then there exists a C^∞ harmonic function η on U such that $\xi(u, v) + \mathrm{i}\,\eta(u, v)$ is holomorphic on U.

Exercise 2-1

Problem (Ex. 1-1)

Let $\xi(u,v) := \log\sqrt{u^2+v^2}$ *be a function defined on* $U := \mathbb{R}^2 \setminus \{(0,0)\}$

1. *Show that* ξ *is harmonic on* U.
2. *Find the conjugate harmonic function* η *of* ξ *on*

$$V = \mathbb{R}^2 \setminus \{(u,0) \,|\, u \leqq 0\} \subset U.$$

3. *Show that there exists no conjugate harmonic function of* ξ *defined on* U.

Exercise 2-2

Problem (Ex. 2-2)

Let $\theta = \theta(u, v)$ *be a smooth function on a domain* $U \subset \mathbb{R}^2$ *such that* $0 < \theta < \pi$*, and set*

$$\Omega := \begin{pmatrix} \theta_u \cot\theta & 0 & \cot\theta \\ -\theta_u \csc\theta & 0 & -\csc\theta \\ 0 & \sin\theta & 0 \end{pmatrix}, \Lambda := \begin{pmatrix} 0 & -\theta_v \csc\theta & -\csc\theta \\ 0 & \theta_v \cot\theta & \cot\theta \\ \sin\theta & 0 & 0 \end{pmatrix}.$$

Prove that the compatibility condition of a system of partial differentail equation

$$\frac{\partial \mathcal{F}}{\partial u} = \mathcal{F}\Omega, \qquad \frac{\partial \mathcal{F}}{\partial v} = \mathcal{F}\Lambda$$

is equivalent to

$$\theta_{uv} = \sin\theta.$$

Exercise 2-3

Problem (Ex. 2-3)

Let $\boldsymbol{v} = \boldsymbol{v}(x, y, z)$ *be a vector field defined on a simply connected domain* U *in* $(\mathbb{R}^3; (x, y, z))$*. Assume that* $\boldsymbol{v}$ *is irrotational, that is,* $\operatorname{rot} \boldsymbol{v} = \mathbf{0}$*. Then there exists a function* $\varphi \colon U \to \mathbb{R}$ *such that* $\boldsymbol{v} = \operatorname{grad} \varphi$*.*

Immersed surfaces

- $p\colon U \to \mathbb{R}^3$: a regular surface
- $\nu\colon U \to \mathbb{R}^3$: the unit normal vector field.

Fundammental forms

$$
\begin{aligned}
ds^2 &:= dp \cdot dp = E\,du^2 + 2F\,du\,dv + G\,dv^2,\\
\widehat{I} &:= \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} {}^t p_u \\ {}^t p_v \end{pmatrix} (p_u, p_v),\\
II &:= -d\nu \cdot dp == L\,du^2 + 2M\,du\,dv + N\,dv^2,\\
\widehat{II} &:= \begin{pmatrix} L & M \\ M & N \end{pmatrix} = -\begin{pmatrix} {}^t p_u \\ {}^t p_v \end{pmatrix} (\nu_u, \nu_v)
\end{aligned}
$$

Curvatures

$$A := \widehat{I}^{-1}\,\widehat{I\!I} = \begin{pmatrix} A^1_1 & A^1_2 \\ A^2_1 & A^2_2 \lambda_1, \lambda_2 \quad \text{: the eigenvalues of } A \end{pmatrix}$$

$$K := \lambda_1\lambda_2 = \det A = \frac{\det\,\widehat{I\!I}}{\det\,\widehat{I}}$$

$$H := \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{2}\operatorname{tr} A.$$