Advanced Topics in Geometry E (MTH.B501)

A review of surface theory

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"Index" formulation

- $(u^1, u^2) = (u, v)$ coordinate number : superscript
- $f_{,i} = \dfrac{\partial f}{\partial u^i}$.

$g_i = \dfrac{\partial}{\partial u^i}$ g_{11} g_{12} g_{21} g_{22}

$E\,du^2 + 2F\,du\,dv + G\,dv^2 = E\,du^2 + F\,du\,dv + F\,dv\,du + G\,dv^2$

$$ds^2 = dp \cdot dp = \sum_{i,j=1}^{2} g_{ij}\, du^i\, du^j, \qquad (g_{ij} := p_{,i} \cdot p_{,j}),$$

subscript.

$g_{ij} = g_{ji}$

$p_u \cdot p_u = p_{,1} \cdot p_{,1}$

$$II = -dp \cdot d\nu = \sum_{i,j=1}^{2} h_{ij}\, du^i\, du^j, \qquad (h_{ij} := -p_{,i} \cdot \nu_{,j} = -p_{,j} \cdot \nu_{,i})$$

$h_{ij} = h_{ji}$

inverse matrix

$$(g^{ij}) := (g_{ij})^{-1} \qquad \sum_k g^{ik} g_{kj} = \delta^i_j$$

Kronecker's delta $= \begin{cases} 1 & i=j \\ 0 & \text{otherwise} \end{cases}$

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \hat{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

$EG - F^2 > 0$

Gauss Frame → (p_u, p_v, ν) Gauss 枠

3×3 matrix-valued function

$$\mathcal{F}\colon U \ni (u^1,u^2) \mapsto \bigl(p_{,1}(u^1,u^2), p_{,2}(u^1,u^2), \nu(u^1,u^2)\bigr) \in \boxed{\mathrm{GL}(3,\mathbb{R})}$$

{the set of 3×3 regular matrices /ℝ

Theorem (Gauss-Weingarten)

$$\mathcal{F}_{,j} \qquad \frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\Omega_j \qquad \left(\Omega_j := \begin{pmatrix} \Gamma^1_{1j} & \Gamma^1_{2j} & -A^1_j \\ \Gamma^2_{1j} & \Gamma^2_{2j} & -A^2_j \\ h_{1j} & h_{2j} & 0 \end{pmatrix}\right)$$

2nd f.f

Weingarten matrix

where

$$\Gamma^k_{ij} := \underbrace{\frac{1}{2}\sum_{l=1}^{2} g^{kl}\bigl(g_{il,j} + g_{lj,i} - g_{ij,l}\bigr)}, \qquad (i,j,k=1,2)$$

Christoffel's symbol

- $\underline{P_{,ij} = \Gamma_{ij}^{1} P_{1} + \Gamma_{ij}^{2} P_{2} + h_{ij}\nu} = \left(\sum_{k=1}^{2} \Gamma_{ij}^{k} P_{k}\right) + h_{ij}\nu$

Gauss - Formula

$\underline{P_{,ij}}$ can be represented by a linear compination of $\underline{P_{,1}, P_{,2}, \nu}$

Set $P_{,ij} = \Lambda_{ij}^{1} P_{,1} + \Lambda_{ij}^{2} P_{,2} + \eta_{ij}\nu$

☆ Since $P_{,j} \cdot \nu = 0$ & $\nu \cdot \nu = 1$

$$P_{,ij} \cdot \nu = \eta_{ij}$$

$$\| \qquad \overset{0}{''}$$

$$(P_{,i} \cdot \nu)_{,j} - P_{i} \cdot \nu_{j} = - P_{i} \cdot \nu_{j} = h_{ij}$$

$\therefore$ $\boxed{\eta_{ij} = h_{ij}}$

- $P_{,ij} = \Lambda^1_{ij} P_{,1} + \Lambda^2_{ij} P_{,2} + \eta_{ij}\nu = \sum_{k=1}^{2} \Lambda^k_{ij} P_{,k} + \eta_{ij}\nu$

$$\underline{P_{,ij}\cdot P_{,l}} = \sum_{k=1}^{2} \Lambda^k_{ij}\left(P_{,k}\cdot P_{,l}\right) = \sum_k \Lambda^k_{ij}\, g_{kl}$$

$$\parallel$$

$$(\underbrace{P_{,i}\cdot P_{,l}}_{g_{il}})_{,j} - \boxed{P_{,i}\cdot P_{,\underset{jl}{lj}}} = g_{il,j} - (P_{,i}\cdot P_{,j})_{,l} + \boxed{P_{,il}\cdot P_{,j}}$$

$$= g_{il,j} - g_{ij,l} + (P_{,l}\cdot P_{,j})_{,i} - P_{,l}\cdot P_{,ji}$$

$$= g_{il,j} - g_{il,l} + g_{lj,i} - \underline{P_{,ij}\cdot P_{,l}}$$

$$\therefore \quad \sum_k \underline{\underline{\Lambda^k_{ij}}}\, g_{kl} = \underline{\frac{1}{2}\left(g_{il,j} + g_{lj,i} - g_{ij,l}\right)}$$

$$\sum_k \underline{\Lambda^k_{ij}}\, g_{kl} = \underline{\tfrac{1}{2}\left(g_{il,j} + g_{lj,i} - g_{ij,l}\right)}$$

$$\sum_l g^{lm} \sum_k \Lambda^k_{ij}\, g_{kl} = \frac{1}{2}\sum_l g^{lm}\left(\cdots \right) = \Gamma^m_{ij}$$

$$\|$$

$$\sum_k \sum_l g^{lm} g_{kl} \Lambda^k_{ij} = \sum_k \overbrace{\boxed{\sum_l g^{ml} g_{lk}}}^{\delta^m_k} \Lambda^k_{ij}$$

$$= \sum_k \delta^m_k \Lambda^k_{ij} = \Lambda^m_{ij}$$

$\square$

$$\nu_{,j} = -A_j^1\, p_1 - A_j^2\, p_2 = -\sum_{k=1}^{2} A_j^k\, p_k$$ Weingarten formula

Set $\nu_{,j} = B_j^1\, p_1 + B_j^2\, p_2 + \cancel{\mu_j\, \nu}$

$$\mu_j = \nu_{,j}\cdot \nu = \frac{1}{2}(\underbrace{\nu\cdot\nu}_{1})_{,j} = 0$$

$$\underset{\displaystyle -h_{jl}}{\underset{\|}{\nu_{,j}\cdot p_{,l}}} = \sum_k B_j^k\, p_{,k}\cdot p_{,l} = \sum_k g_{kl}\, B_j^k$$

$$\dashrightarrow \quad B_j^m = -\sum g^{lm}\, h_{jl} = -A_j^m \quad \square$$

Adapted Frame

- $p\colon U \to \mathbb{R}^3$: a regular surface
- ν: the unit normal
- $\mathcal{E} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) : U \to \mathrm{SO}(3)$, ($\boldsymbol{e}_3 = \nu$): an adapted frame

orthonormal.

ν

$$\check{I} = \begin{pmatrix} g^1_1 & g^1_2 \\ g^2_1 & g^2_2 \end{pmatrix} \quad \text{such that} \quad (p_u, p_v) = (\boldsymbol{e}_1, \boldsymbol{e}_2)\,\check{I}$$

$$\check{II} = \begin{pmatrix} h^1_1 & h^1_2 \\ h^2_1 & h^2_2 \end{pmatrix} \quad \text{such that} \quad \big((\boldsymbol{e}_3)_u, (\boldsymbol{e}_3)_v\big) = -(\boldsymbol{e}_1, \boldsymbol{e}_2)\,\check{II}\,.$$

Gauss-Weingarten formula

$\mathcal{E} \in SO(3)$

$$\mathcal{E}_u = \mathcal{E}\Omega, \qquad \mathcal{E}_v = \mathcal{E}\Lambda$$

$$\left(\Omega := \begin{pmatrix} 0 & -\alpha & -h_1^1 \\ \alpha & 0 & -h_1^2 \\ h_1^1 & h_1^2 & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} 0 & -\beta & -h_2^1 \\ \beta & 0 & -h_2^2 \\ h_2^1 & h_2^2 & 0 \end{pmatrix}\right).$$

skew-symmetric matrices

Exercise 3-1

Problem (Ex. 3-1)

Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ *are given in the form*

$g_{11} = e^{2\sigma}$ $g_{12} = g_{21} = 0$ $g_{22} = e^{2\sigma}$

$$\underline{ds^2 = e^{2\sigma}((du^1)^2 + (du^2)^2)}, \qquad II = \sum_{i,j=1}^{2} h_{ij}\, du^i\, du^j,$$

where σ *is a smooth function in* (u^1, u^2). *Compute the matrices* Ω_j $(j = 1, 2)$. (3.17)

Exercise 3-1 2

Problem (Ex. 3-2)

Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ *are given in the form*

$h_{11} = h_{22} = 0$

$h_{12} = h_{21} = \sin\theta$

$$ds^2 = (du^1)^2 + 2\cos\theta\, du^1\, du^2 + (du^2)^2, \qquad II = 2\sin\theta\, du^1\, du^2,$$

where θ *is a smooth function in* (u^1, u^2)*. Compute the matrices* Ω_j $(j = 1, 2)$