Advanced Topics in Geometry E (MTH.B501)

A review of surface theory

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“Index” formulation

- $(u^1, u^2) = (u, v)$
- $f_{,i} = \dfrac{\partial f}{\partial u^i}$

$$ds^2 = dp \cdot dp = \sum_{i,j=1}^{2} g_{ij}\, du^i\, du^j, \qquad (g_{ij} := p_{,i} \cdot p_{,j}),$$

$$II = -dp \cdot d\nu = \sum_{i,j=1}^{2} h_{ij}\, du^i\, du^j, \qquad (h_{ij} := -p_{,i} \cdot \nu_{,j} = -p_{,j} \cdot \nu_{,i})$$

$$(g^{ij}) := (g_{ij})^{-1} \qquad \sum_{k} g^{ik} g_{kj} = \delta^i_j$$

Gauss Frame

$$\mathcal{F}\colon U \ni (u^1,u^2) \mapsto \big(p_{,1}(u^1,u^2), p_{,2}(u^1,u^2), \nu(u^1,u^2)\big) \in \mathrm{GL}(3,\mathbb{R})$$

Theorem

$$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\Omega_j \qquad \left(\Omega_j := \begin{pmatrix} \Gamma^1_{1j} & \Gamma^1_{2j} & -A^1_j \\ \Gamma^2_{1j} & \Gamma^2_{2j} & -A^2_j \\ h_{1j} & h_{2j} & 0 \end{pmatrix}\right)$$

where

$$\Gamma^k_{ij} := \frac{1}{2}\sum_{l=1}^{2} g^{kl}(g_{il,j} + g_{lj,i} - g_{ij,l}), \qquad (i,j,k=1,2)$$

Adapted Frame

- $p\colon U \to \mathbb{R}^3$: a regular surface
- ν: the unit normal
- $\mathcal{E} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) : U \to \mathrm{SO}(3)$, $(\boldsymbol{e}_3 = \nu)$: an adapted frame

$$\check{I} = \begin{pmatrix} g^1_1 & g^1_2 \\ g^2_1 & g^2_2 \end{pmatrix} \qquad \text{such that} \qquad (p_u, p_v) = (\boldsymbol{e}_1, \boldsymbol{e}_2)\,\check{I}$$

$$\check{II} = \begin{pmatrix} h^1_1 & h^1_2 \\ h^2_1 & h^2_2 \end{pmatrix} \qquad \text{such that} \qquad \big((\boldsymbol{e}_3)_u, (\boldsymbol{e}_3)_v\big) = -(\boldsymbol{e}_1, \boldsymbol{e}_2)\,\check{II}\,.$$

Gauss-Weingarten formula

$$\mathcal{E}_u = \mathcal{E}\Omega, \qquad \mathcal{E}_v = \mathcal{E}\Lambda$$

$$\left(\Omega := \begin{pmatrix} 0 & -\alpha & -h_1^1 \\ \alpha & 0 & -h_1^2 \\ h_1^1 & h_1^2 & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} 0 & -\beta & -h_2^1 \\ \beta & 0 & -h_2^2 \\ h_2^1 & h_2^2 & 0 \end{pmatrix}\right).$$

Exercise 3-1

Problem (Ex. 3-1)

Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ *are given in the form*

$$ds^2 = e^{2\sigma}((du^1)^2 + (du^2)^2), \qquad II = \sum_{i,j=1}^{2} h_{ij}\, du^i\, du^j,$$

where σ *is a smooth function in* (u^1, u^2)*. Compute the matrices* Ω_j *(*$j = 1, 2$*).*

Exercise 3-1

Problem (Ex. 3-2)

Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ *are given in the form*

$$ds^2 = (du^1)^2 + 2\cos\theta \, du^1 \, du^2 + (du^2)^2, \qquad II = 2\sin\theta \, du^1 \, du^2,$$

where θ *is a smooth function in* (u^1, u^2). *Compute the matrices* Ω_j *(*$j = 1, 2$*)*