Advanced Topics in Geometry E (MTH.B501)

The Gauss and Codazzi equations

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The Gauss-Weingarten formulas

$$p = p(u^1, u^2) \text{ : a parametrized surface} \quad \text{(regular)}$$

$$\nu = \nu(u^1, u^2) \text{ : the unit normal vector field}$$

$$\mathcal{F} = (\underbrace{p_{,1}, p_{,2}}_{\text{(lin indep)}}, \nu) \text{ : the Gauss Frame}$$

$$\mathcal{F}(u^1, u^2) \in GL(3, \mathbb{R})$$

The Gauss-Weingarten formula:

$$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\Omega_j \qquad (j = 1, 2)$$

Ω_j : 3×3 matrix-valued function

The Gauss-Weingarten formulas

$\mathcal{F}_{,j} = \mathcal{F}\Omega_j$

$$\Omega_j = \begin{pmatrix} \Gamma^1_{j1} & \Gamma^1_{j2} & -A^1_j \\ \Gamma^2_{j1} & \Gamma^2_{j2} & -A^2_j \\ h_{j1} & h_{j2} & 0 \end{pmatrix}$$

$$\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{kl}(g_{lj,i} + g_{il,j} - g_{ij,l})$$

Christoffel's symbols

$g_{ji} = g_{ij} = p_{,i} \cdot p_{,j}, \qquad (g^{ij}) = (g_{ij})^{-1}$

$h_{ji} = h_{ij} = -p_{,i} \cdot \nu_{,j} = -\nu_{,i} \cdot p_{,j}$

2nd f.f. form

$$A^i_j = \sum_l g^{il} h_{lj}$$

Weingarten matrix

straightforward

Problem (Ex. 3-1)

Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ are given in the form

isothermal parameter.

$$ds^2 = e^{2\sigma}((du^1)^2 + (du^2)^2), \qquad II = \sum_{i,j=1}^{2} h_{ij}\, du^i\, du^j,$$

$$g_{ij} = e^{2\sigma}\delta_{ij}$$

where σ is a smooth function in (u^1, u^2). Compute the matrices Ω_j $(j = 1, 2)$.

$$\Omega_1 = \begin{pmatrix} \sigma_{,1} & \sigma_{,2} & -e^{-2\sigma}h_{11} \\ -\sigma_{,2} & \sigma_{,1} & -e^{-2\sigma}h_{12} \\ h_{11} & h_{12} & 0 \end{pmatrix} \quad \Omega_2 = \begin{pmatrix} \sigma_{,2} & -\sigma_{,1} & -e^{-2\sigma}h_{21} \\ \sigma_{,1} & \sigma_{,2} & -e^{-2\sigma}h_{22} \\ h_{21} & h_{22} & 0 \end{pmatrix}$$

Problem (Ex. 3-2)

Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ *are given in the form* $g_{11} = g_{22} = 1$ $h_{11} = h_{22} = 0$ $h_{12} = \sin\theta$

$$ds^2 = (du^1)^2 + 2\cos\theta\, du^1\, du^2 + (du^2)^2, \quad g_{12} = \cos\theta \quad II = 2\sin\theta\, du^1\, du^2,$$

where θ *is a smooth function in* (u^1, u^2)*. Compute the matrices* Ω_j $(j = 1, 2)$

Ω_j : $\Omega_1 = \Omega$ in Ex 2-2

$\Omega_2 = \Lambda$

asymptotic

$$K = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2} = -1 = \text{const.}$$

Chebyshev net.