Advanced Topics in Geometry E (MTH.B501)

The Gauss and Codazzi equations

Kotaro Yamada
kotaro@math.titech.ac.jp
http://www.math.titech.ac.jp/~kotaro/class/2022/geom-e/

Tokyo Institute of Technology

2022/05/17

The Gauss-Weingarten formulas

$p = p(u^1, u^2)$: a parametrized surface

$\nu = \nu(u^1, u^2)$: the unit normal vector field

$\mathcal{F} = (p_{,1}, p_{,2}, \nu)$: the Gauss Frame

The Gauss-Weingarten formula:

$$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\Omega_j \qquad (j = 1, 2)$$

The Gauss-Weingarten formulas

$$\Omega_j = \begin{pmatrix} \Gamma^1_{j1} & \Gamma^1_{j2} & -A^1_j \\ \Gamma^2_{j1} & \Gamma^2_{j2} & -A^2_j \\ h_{j1} & h_{j2} & 0 \end{pmatrix}$$

$$\Gamma^k_{ij} = \frac{1}{2} \sum_l g^{kl} (g_{lj,i} + g_{il,j} - g_{ij,l})$$

$$g_{ij} = p_{,i} \cdot p_{,j}, \qquad (g^{ij}) = (g_{ij})^{-1}$$

$$h_{ij} = -p_{,i} \cdot \nu_{,j} = -\nu_{,i} \cdot p_{,j}$$

$$A^i_j = \sum_l g^{il} h_{lj}$$

Exercise 3-1

Problem (Ex. 3-1)

Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ *are given in the form*

$$ds^2 = e^{2\sigma}((du^1)^2 + (du^2)^2), \qquad II = \sum_{i,j=1}^{2} h_{ij}\, du^i\, du^j,$$

where σ *is a smooth function in* (u^1, u^2)*. Compute the matrices* Ω_j $(j = 1, 2)$*.*

Exercise 3-2

Problem (Ex. 3-2)

Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ *are given in the form*

$$ds^2 = (du^1)^2 + 2\cos\theta\, du^1\, du^2 + (du^2)^2, \qquad II = 2\sin\theta\, du^1\, du^2,$$

where θ *is a smooth function in* (u^1, u^2)*. Compute the matrices* Ω_j $(j = 1, 2)$