Advanced Topics in Geometry E (MTH.B501)

The $\underbrace{\text{Gauss}}_{1}$ and $\underbrace{\text{Codazzi}}_{2}$ equations

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The ingetrablility conditions

The Gauss-Weingarten formulas: expressed in terms of g_{ij}, h_{ij}.

the Gauss frame

$$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\Omega_j, \qquad \Omega_j = \begin{pmatrix} \Gamma^1_{j1} & \Gamma^1_{j2} & -A^1_j \\ \Gamma^2_{j1} & \Gamma^2_{j2} & -A^2_j \\ h_{j1} & h_{j2} & 0 \end{pmatrix} \qquad (j = 1, 2)$$

The integrability conditions: g_{ij}, h_{ij} must satisfy:

$$\boxed{\frac{\partial \Omega_1}{\partial u^2} - \frac{\partial \Omega_2}{\partial u^1} - \Omega_1\Omega_2 + \Omega_2\Omega_1 = O}$$

9 equalities reduced to 3

$$\left\{\begin{array}{l} \dfrac{\partial \mathcal{F}}{\partial u^1} = \mathcal{F}\Omega_1 \Rightarrow \dfrac{\partial^2 \mathcal{F}}{\partial u^2 \partial u^1} = \dfrac{\partial \mathcal{F}}{\partial u^2}\Omega_1 + \mathcal{F}\dfrac{\partial \Omega_1}{\partial u^2} = \mathcal{F}\left(\Omega_2\Omega_1 + \dfrac{\partial \Omega_1}{\partial u^2}\right) \\ \dfrac{\partial \mathcal{F}}{\partial u^2} = \mathcal{F}\Omega_2 \qquad \dfrac{\partial^2 \mathcal{F}}{\partial u^1 \partial u^2} = \mathcal{F}\left(\Omega_1\Omega_2 + \dfrac{\partial \Omega_2}{\partial u^1}\right) \end{array}\right.$$

The Gauss and Codazzi equations

Theorem (Theorem 4.3)

The integrability condition of G-W formula is equivalent to the following three equalities:

$$h_{11,2} - h_{21,1} = \sum_j \left(\Gamma^j_{21} h_{1j} - \Gamma^j_{11} h_{2j}\right)$$

$$h_{12,2} - h_{22,1} = \sum_j \left(\Gamma^j_{22} h_{1j} - \Gamma^j_{12} h_{2j}\right)$$

Codazzi equations.

Gaussian curv

$$K_{ds^2} = \frac{1}{g}(h_{11}h_{22} - h_{12}h_{21}) \left(= K\right).$$

Gauss eq.

$g = g_{11}g_{22} - g_{12}g_{21} \ (>0)$

Theorem (Theorem 4.3, continued)

Here, $g := \det(g_{ij}) = g_{11}g_{22} - g_{12}g_{21}$, *and* 断面曲率.

$$K_{ds^2} := \frac{1}{g} R_{12},$$

sectional curvature of ds^2 expressed in terms of g_{ij}

$$\begin{aligned} R_{jk} := {} & \frac{1}{2}(g_{1k,2j} - g_{1j,2k} + g_{2j,1k} - g_{2k,1j}) \\ & - \sum_{i,s} g_{is}(\Gamma^s_{ks}\Gamma^i_{1j} - \Gamma^s_{k1}\Gamma^i_{2j}) \\ & + 2\sum_{l,s} g_{kl}(\Gamma^l_{s2}\Gamma^s_{1j} - \Gamma^l_{1s}\Gamma^s_{2j}). \end{aligned}$$

Formulas

$$(A^{-1})' = -A^{-1}A'A^{-1}$$

- $(g^{ij}) = (g_{ij})^{-1}$ $\quad g^{ij} = g^{ji}$

$AA^{-1} = id$

$A'A^{-1} + A(A^{-1})' = 0$

$$\sum_l g^{il} g_{lj} = \delta^i_j, \qquad g^{il}_{,k} = -\sum_{\alpha,\beta} g^{\alpha i} g^{\beta l} g_{\alpha\beta,k}$$

definition

$\sum g_{mk} g^{kl} = \delta^l_m$

- $\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{kl}(g_{lj,i} + g_{il,j} - g_{ij,l}) = \Gamma^k_{ji}$ $\Rightarrow \sum_k g_{mk}\Gamma^k_{ij}$

def.

$= \frac{1}{2}(g_{mj,i} + g_{im,j} - g_{ij,m})$

$$g_{ij,k} = \sum_l (g_{il}\Gamma^l_{jk} + g_{lj}\Gamma^l_{ik}), \quad \sum_i \Gamma^i_{ji} = \frac{1}{2g} g_{,j} \quad (g = \det(g_{ij}))$$

- $A^i_j = \sum_l g^{il} h_{lj}$

def.

Proof of Theorem 4.3

Codazzi

$$\begin{pmatrix} I_1^1 & I_2^1 & I_3^1 \\ I_1^2 & I_2^2 & I_3^2 \\ I_1^3 & I_2^3 & \cancel{I_3^3} \end{pmatrix} := \Omega_{1,2} - \Omega_{2,1} - \Omega_1\Omega_2 + \Omega_2\Omega_1 = 0$$

(Codazzi)

$$\Omega_j = \begin{pmatrix} \Gamma_{j1}^1 & \Gamma_{j2}^1 & -A_j^1 \\ \Gamma_{j1}^2 & \Gamma_{j2}^2 & -A_j^2 \\ h_{j1} & h_{j2} & 0 \end{pmatrix} \qquad (j = 1, 2)$$

$$I^3_2 = h_{11} A^1_2 + h_{12} A^2_1 - h_{21} A^1_1 - h_{22} A^2_2$$

$$= \sum_j h_{1j} A^j_2 - \sum_j h_{2j} A^j_1$$

$$= \sum_j \left(h_{1j} \sum_l g^{jl} h_{l2} \right) - \sum_j \left(h_{2j} \sum_l g^{jl} h_{l1} \right)$$

$$= \sum_{j,l} \left(g^{jl} h_{1j} h_{l2} \right) - \sum_{\substack{j,l \\ l\ j}} \left(g^{\overset{lj}{jl}} h_{\underset{j}{l}1} h_{2\underset{l}{j}} \right)$$

$$= \sum_{j,l} g^{jl} \left(h_{1j} h_{l2} - h_{j1} h_{2l} \right) = 0$$

$I^3_3 = 0$ Automatic.

$$I^3_k = h_{1k,2} - h_{2k,1} - \sum_l h_{1l}\Gamma^l_{2k} + \sum_l h_{2l}\Gamma^l_{1k} .$$

$(k=1,2)$

Codazzi equations

$$I^k_3 = -A^k_{1,2} + A^k_{2,1} + \sum_l \Gamma^k_{l1} A^l_2 - \sum_l \Gamma^k_{l2} A^l_1 .$$

$$\sum_k g_{km} I^k_3 = -\sum_k g_{km}\Big(\sum_l g^{kl} h_{l1}\Big)_{,2} + \sum_k g_{km}\Big(\sum_l g^{kl} h_{l2}\Big)_{,1}$$

$$+ \sum_l \sum_s \Gamma^k_{l1} g^{ls} h_{s2} - \sum_l \sum_s \Gamma^k_{l2} g^{ls} h_{s1}$$

$$= - I^3_m$$

$(m=1,2)$

$$\begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} I^1_3 \\ I^2_3 \end{pmatrix} = - \begin{pmatrix} I^3_1 \\ I^3_2 \end{pmatrix}$$

$$\boxed{\begin{array}{c} I^1_3 = I^2_3 = 0 \\ \Leftrightarrow I^3_1 = I^3_2 = 0 \end{array}}$$

$$\sum_{i=1}^{2} g_{ik} I^i_j = \frac{1}{2}(g_{1k,2j} - g_{1j,2k} + g_{2j,1k} - g_{2k,1j})$$
$$- \sum_{i,s} g_{is}(\Gamma^s_{ks}\Gamma^i_{1j} - \Gamma^s_{k1}\Gamma^i_{2j})$$
$$+ 2\sum_{l,s} g_{kl}(\Gamma^l_{s2}\Gamma^s_{1j} - \Gamma^l_{1s}\Gamma^s_{2j}) - \boxed{h_{1k}h_{2j} - h_{2k}h_{1j}}$$

$i, j = 1, 2$

!!

I_{kj}

R_{kj}

skew symmetric in k, j

skew symmetric in k, j

expand in terms of $g_{\alpha\beta,\gamma}$ $\Rightarrow$ skew symmetricity in k, j

$I_{kj} = 0$ $(j, k = 1 .. 2)$

$\Leftrightarrow$ $I_{12} = 0$

Exercise 4-1

Problem (Ex. 4-1)

Assume $L = N = 0$*, that is,* $II = 2M\,du\,dv = 2h_{12}\,du^1\,du^2$*, Prove that, if the Gaussian curvature* K *is negative constant,*

$$E_v = G_u = 0, \qquad \text{that is, } g_{11,2} = g_{22,1} = 0.$$

$$\mathrm{II} = 2M\,du\,dv \qquad ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

$$\det \hat{\mathrm{II}} = \det\begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix} = -M^2 \leq 0 \quad \therefore (K \leq 0)$$

Assume: $K = -c^2 \quad (c > 0 : \text{const})$

(Codazzi)

Exercise 4-2

Problem (Ex. 4-2)

Assume $F = 0$ and $E = G = e^{2\sigma}$, where σ is a function in (u, v). Let $z = u + iv$ ($i = \sqrt{-1}$) and define a complex-valued function q in z by

$$q(z) := \frac{L(u,v) - N(u,v)}{2} - iM(u,v).$$

Prove that the Codazzi equations are equivalent to

$$\frac{\partial q}{\partial \bar{z}} = e^{2\sigma} \frac{\partial H}{\partial z},$$

where H is the mean curvature, and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

- $ds^2 = e^{2\sigma}(du^2 + dv^2) = e^{2\sigma}\, dz\, d\bar{z}$

(u,v): isothermal.

$z = u + iv$
$dz = du + i\,dv$
$d\bar{z} = du - i\,dv$

- $\mathrm{II} = L\,du^2 + 2M\,du\,dv + N\,dv^2$

$q := \frac{L-N}{2} - iM$

$\boxed{H = \frac{1}{2} e^{-2\sigma}(L+N)}$

$q : z \mapsto q(z)$

- Codazzi $\quad \frac{\partial q}{\partial \bar{z}} = e^{2\sigma} \frac{\partial H}{\partial z}$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right) \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$$

Cauchy-Riemann operator