Advanced Topics in Geometry E (MTH.B501)

The Fundamental Theorem for Surfaces

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$p = p(u^1, u^2)$: a parametrized surface in $\mathbb{R}^3$

$\nu = \nu(u^1, u^2)$: the unit normal vector field.

$\mathcal{F} = (p_{,1}, p_{,2}, \nu)$: the Gauss Frame

Rem. By taking

$$\nu = \frac{p_{,1} \times p_{,2}}{|p_{,1} \times p_{,2}|}$$

as a unit normal we may assume

$$\boxed{\det \mathcal{F} > 0}$$

$$\det \mathcal{F} = (p_{,1} \times p_{,2}) \cdot \nu = |p_{,1} \times p_{,2}| > 0$$

The Gauss-Weingarten formula:

$$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\Omega_j \qquad (j = 1, 2)$$

The Gauss and Codazzi equations

The integrability for ($\mathcal{F}_{,i} = \mathcal{F}\Omega_i$)

$$h_{11,2} - h_{21,1} = \sum_j \left(\Gamma^j_{21} h_{1j} - \Gamma^j_{11} h_{2j}\right)$$

Codazzi

$$h_{12,2} - h_{22,1} = \sum_j \left(\Gamma^j_{22} h_{1j} - \Gamma^j_{12} h_{2j}\right)$$

$$K_{ds^2} = \frac{1}{g}(h_{11}h_{22} - h_{12}h_{21})\big(= K\big)$$

Gauss

Problem (Ex. 4-1)

Assume $L = N = 0$, that is, $II = 2M\,du\,dv = 2h_{12}\,du^1\,du^2$, Prove that, if the Gaussian curvature K is negative constant,

codazzi

$$E_v = G_u = 0, \qquad \text{that is,} g_{11,2} = g_{22,1} = 0.$$

$$\mathrm{II} = 2M\,du\,dv = 2h_{12}\,du^1 du^2$$

漸近座標系

asymptotic coordinate system

$$K = -c^2 \quad (c>0,\ \text{const})$$

$$\| $$

$$\frac{\det(h_{ij})}{\det(g_{ij})} = -h_{12}^2/g \qquad g := g_{11}g_{22} - g_{12}^2$$

$$\cancel{h_{11,2}} - h_{21,1} = \sum_j \left(\Gamma^j_{21} h_{1j} - \Gamma^j_{11} h_{2j}\right) \quad (1) \qquad -h_{12,1} = \sum \left(\Gamma^j_{21} h_{1j} - \Gamma^j_{11} h_{2j}\right)$$

$$h_{12,2} - \cancel{h_{22,1}} = \sum_j \left(\Gamma^j_{22} h_{1j} - \Gamma^j_{12} h_{2j}\right) \quad (2) \qquad h_{12,2} = \sum \left(\Gamma^j_{22} h_{1j} - \Gamma^j_{12} h_{2j}\right)$$

$$h_{12}^2 = + c^2 \cdot g \qquad h_{12} = c\sqrt{g} \quad (\pm !)$$

$$-c(\sqrt{g})_{,1} = \cancel{\Gamma^1_{21}} h_{11} + \Gamma^2_{21} h_{12} - \Gamma^1_{11} h_{21} - \cancel{\Gamma^2_{11}} h_{22}$$

$$= c\sqrt{g}\left(\Gamma^2_{21} - \Gamma^1_{11}\right)$$

$$(\sqrt{g})_{,1} = (\sqrt{g})\left(\Gamma^1_{11} + \Gamma^2_{12}\right) \qquad (3.25)$$

$$-\cancel{\Gamma^1_{11}} - \Gamma^2_{12} = \Gamma^2_{21} - \cancel{\Gamma^1_{11}} \qquad \underline{\Gamma^2_{21} = 0}$$

$$\rightsquigarrow$$

Exercise 4-2

Problem (Ex. 4-2)

Assume $F = 0$ and $E = G = e^{2\sigma}$, where σ is a function in (u, v). Let $z = u + iv$ ($i = \sqrt{-1}$) and define a complex-valued function q in z by

$$q(z) := \frac{L(u,v) - N(u,v)}{2} - iM(u,v).$$

Prove that the Codazzi equations are equivalent to

$$\frac{\partial q}{\partial \bar{z}} = e^{2\sigma}\frac{\partial H}{\partial z},$$

where H is the mean curvature, and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

- $ds^2 = e^{2\sigma}(du^2 + dv^2)$ isothermal coordinates.

$z = u + iv$

$= e^{2\sigma}\, dz\, d\bar{z}$

$= e^{2\sigma}\, |dz|^2$

- $q := \dfrac{L-N}{2} - iM = q(z)$

Codazzi

$$\Leftrightarrow \frac{\partial q}{\partial \bar{z}} = e^{2\sigma} \frac{\partial H}{\partial z}$$

Rem

- $\exists$ isothermal coord. on 2-dim Riem mfd.
- a parameter change between isothermal coordinate systems :

$$\begin{cases} u = u(\xi, \eta) \\ v = v(\xi, \eta) \end{cases}$$

$$\begin{cases} u_\xi = \pm v_\eta \\ u_\eta = \mp v_\xi \end{cases}$$

$\xi + i\eta \mapsto u + iv$

- holomorphic or anti holomorphic

$$\mathrm{II} = L\,du^2 + 2M\,du\,dv + N\,dv^2$$

$dz = du + i\,dv$

$d\bar{z} = du - i\,dv$

$du = \frac{1}{2}(dz + d\bar{z})$

$dv = -\frac{i}{2}(dz - d\bar{z})$

$$= L\,\frac{1}{4}\left(dz^2 + 2\,dz\,d\bar{z} + d\bar{z}^2\right)$$

(2,0)-part

$$- 2M\,\frac{i}{4}\left(dz^2 - d\bar{z}^2\right)$$

$$- N\cdot\frac{1}{4}\left(dz^2 + 2\,dz\,d\bar{z} + d\bar{z}^2\right)$$

Hopf diff.

$$= \boxed{\frac{1}{4}\left((L-N) - 2iM\right)dz^2}$$

$$\frac{1}{4}\left((L-N) + 2iM\right)d\bar{z}^2$$

$$+ \boxed{\frac{1}{4}(L+N)\,dz\,d\bar{z}} = \frac{1}{2}H\,ds^2$$

$$\frac{\partial q}{\partial \bar{z}} = e^{2v} \frac{\partial H}{\partial z} \quad : \quad H : \text{const} \Rightarrow \frac{\partial q}{\partial \bar{z}} = 0 \quad ⊛$$

⊛ q is a holomorphic function in z

use of

$\rightarrow$ complex function theory.