Advanced Topics in Geometry E (MTH.B501)

The Fundamental Theorem for Surfaces

Kotaro Yamada
kotaro@math.titech.ac.jp
http://www.math.titech.ac.jp/~kotaro/class/2022/geom-e/

Tokyo Institute of Technology

2022/05/24

The Gauss-Weingarten formulas

$$p = p(u^1, u^2) \text{ : a parametrized surface}$$
$$\nu = \nu(u^1, u^2) \text{ : the unit normal vector field}$$
$$\mathcal{F} = (p_{,1}, p_{,2}, \nu) \text{ : the Gauss Frame}$$

The Gauss-Weingarten formula:

$$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\Omega_j \qquad (j = 1, 2)$$

The Gauss and Codazzi equations

$$h_{11,2} - h_{21,1} = \sum_j \left(\Gamma^j_{21} h_{1j} - \Gamma^j_{11} h_{2j} \right)$$

$$h_{12,2} - h_{22,1} = \sum_j \left(\Gamma^j_{22} h_{1j} - \Gamma^j_{12} h_{2j} \right)$$

$$K_{ds^2} = \frac{1}{g}(h_{11}h_{22} - h_{12}h_{21})\big(= K\big)$$

Exercise 4-1

Problem (Ex. 4-1)

Assume $L = N = 0$*, that is,* $II = 2M\,du\,dv = 2h_{12}\,du^1\,du^2$*, Prove that, if the Gaussian curvature* K *is negative constant,*

$$E_v = G_u = 0, \qquad \textit{that is,} g_{11,2} = g_{22,1} = 0.$$

Exercise 4-2

Problem (Ex. 4-2)

Assume $F = 0$ and $E = G = e^{2\sigma}$, where σ is a function in (u, v). Let $z = u + iv$ ($i = \sqrt{-1}$) and define a complex-valued function q in z by

$$q(z) := \frac{L(u,v) - N(u,v)}{2} - iM(u,v).$$

Prove that the Codazzi equations are equivalent to

$$\frac{\partial q}{\partial \bar{z}} = e^{2\sigma} \frac{\partial H}{\partial z},$$

where H is the mean curvature, and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$