Advanced Topics in Geometry E (MTH.B501)

The Fundamental Theorem for Surfaces

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2022/05/24

Given data: six functions defined on $U \subset \mathbb{R}^2$.

$$g_{12} = g_{21}$$
$$h_{12} = h_{21}$$

$$\widehat{I} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \qquad \widehat{II} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix},$$

$$(g^{ij}) = (g_{ij})^{-1}$$

Assumption:

$$g_{11} > 0, \qquad g_{22} > 0, \quad \text{and} \quad \underbrace{g_{11}g_{22} - g_{12}g_{21}}_{} > 0$$

$$g = g_{11}g_{22} - g_{12}g_{21}$$

Set up:

✓ $$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{2} g^{kl}(g_{lj,k} + g_{il,j} - g_{jl,k}), \qquad A^i_j = \sum_{l=1}^{2} \cancel{g_{jl}h_{il}}$$

$$\sum_l g^{il} h_{lj}$$

The statement

Theorem (Theorem 5.1)

Assume U is simply connected, and (g_{ij}) and (h_{ij}) satisfy the Gauss equation and Codazzi equations. Then there exists a regular surface $p \colon U \to \mathbb{R}^3$ such that

- *the first fundamental form of p is $ds^2 = \sum_{i,j} g_{ij} du^i\, du^j$,*
- *the second fundamental form of p with respect to the unit normal vector field $\nu := (p_{,1} \times p_{,2})/|p_{,1} \times p_{,2}|$ coincides with $II = \sum_{i,j} h_{ij} du^i\, du^j$.*

Moreover, such a surface p is unique up to a transformation

$$p \mapsto Rp + \boldsymbol{a}, \qquad R \in \mathrm{SO}(3),\ \boldsymbol{a} \in \mathbb{R}^3.$$

|| the orthogonal matrices with det = +1

rotation

Uniqueness

$p, \tilde{p} : U \to \mathbb{R}^3$ surfaces with common (g_{ij}) (h_{ij})

$\mathcal{F} = (p_{,1}, p_{,2}, \nu)$ $\quad \tilde{\mathcal{F}} = (\tilde{p}_{,1}, \tilde{p}_{,2}, \tilde{\nu})$: the Gauss frames.

common Gauss Weingarten

$\Rightarrow \mathcal{F}_j = \mathcal{F}\Omega_j \qquad \tilde{\mathcal{F}}_j = \tilde{\mathcal{F}}\Omega_j$

$$\Omega_j = \begin{pmatrix} \Gamma^1_{1j} & \Gamma^1_{2j} & -A^1_j \\ \Gamma^2_{1j} & \Gamma^2_{2j} & -A^2_j \\ h_{1j} & h_{2j} & 0 \end{pmatrix}$$

$\Rightarrow \tilde{\mathcal{F}} = {}^{\exists} R \mathcal{F}$

R: 3×3 matrix

$(\tilde{\mathcal{F}}\mathcal{F}^{-1})_j = \tilde{\mathcal{F}}_j \mathcal{F}^{-1} + \tilde{\mathcal{F}}\,\mathcal{F}^{-1}_j$ $\quad (A^{-1})' = -A^{-1}A'A^{-1}$ $\quad AA^{-1} = \mathrm{id}$

$= \underbrace{\tilde{\mathcal{F}}_j}_{\tilde{\mathcal{F}}\Omega_j} \mathcal{F}^{-1} - \tilde{\mathcal{F}}\mathcal{F}^{-1}\underbrace{\mathcal{F}_j}_{\mathcal{F}\Omega_j}\mathcal{F}^{-1} = \tilde{\mathcal{F}}(\Omega_j - \Omega_j)\mathcal{F}^{-1} = 0$

$\Rightarrow \tilde{\mathcal{F}}\mathcal{F}^{-1} = R$: const

$\hat{\mathcal{F}} = R\mathcal{F}$; $R \in SO(3)$

$\because$

$$
{}^t\hat{\mathcal{F}}\hat{\mathcal{F}} = \begin{pmatrix} {}^t\tilde{p}_{,1} \\ {}^t\tilde{p}_{,2} \\ {}^t\tilde{\nu} \end{pmatrix} (\tilde{p}_{,1}, \tilde{p}_{,2}, \tilde{\nu}) = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
\| \qquad\qquad\qquad\qquad\qquad\qquad = {}^t\mathcal{F}\mathcal{F}
$$

$$
{}^t\mathcal{F}\,{}^tRR\mathcal{F}
$$

$\therefore$ ${}^tRR = id$ i.e $R \in O(3)$

As we assumed

- $\underline{\det \mathcal{F} > 0 \ \& \ \det \hat{\mathcal{F}} > 0}$, $\det R > 0$.

$\therefore$ $R \in SO(3)$

$$\tilde{\mathcal{F}} = R\mathcal{F} \qquad R \in SO(3)$$

$$\tilde{p}_{,1} = R p_{,1}, \qquad \tilde{p}_{,2} = R p_{,2}$$

$$\Rightarrow (\tilde{p} - Rp)_{,j} = 0 \qquad \therefore \tilde{p} = Rp + \underset{\text{const}}{a}$$

$\square$

Existence

Gauss - Codazzi $\Rightarrow$ "$\mathcal{F}_j = \mathcal{F}\Omega_j$" : integrable

- <u>Initial conditions</u> $P_0 \in U$: fix

<u>Take</u>

$$\left.\begin{aligned} \mathcal{F}_0 &:= (v_1, v_2, v_3) \\ v_i \cdot v_j &= g_{ij}(P_0) \qquad (i, j = 1, 2) \\ v_i \cdot v_3 &= 0 \\ v_3 \cdot v_3 &= 1 \end{aligned}\right\}$$

- <u>Solve</u> $\mathcal{F}_j = \mathcal{F}\Omega_j$ with $\mathcal{F}(P_0) = \mathcal{F}_0$

We get $\mathcal{F} = (a_1, a_2, a_3) : U \longrightarrow GL(3, \mathbb{R})$

$p_{,1}$ $p_{,2}$

$\mathcal{F}_2 = \mathcal{F}\Omega_2$

$$(a_1)_{,2} = \Gamma^1_{12} a_1 + \Gamma^2_{12} a_2 + h_{12} a_3$$
$$(a_2)_{,1} = \Gamma^1_{21} a_1 + \Gamma^2_{21} a_2 + h_{21} a_3$$

i.e. $\omega = a_1 \, du^1 + a_2 \, du^2$

satisfies $d\omega = 0$

$\Longrightarrow$ Poincaré lemma

$\exists\, p : U \longrightarrow \mathbb{R}^3$

s.t. $dp = \omega$ $\begin{cases} p_{,1} = a_1 \\ p_{,2} = a_2 \end{cases}$

$$\begin{cases} \mathcal{F} = (a_1, a_2, a_3) & \mathcal{F}_{,j} = \mathcal{F}\Omega_j \\ p: \quad p_{,1} = a_1 \quad p_{,2} = a_2 \end{cases}$$

<u>p is the desired one</u>

$$\left(\begin{array}{l} g_{ij} = p_{,i} \cdot p_{,j} \\ h_{ij} = p_{,ij} \cdot \nu \end{array} \right)$$

- $\left({}^t\mathcal{F}\mathcal{F}\right)_{,j}$

$$= {}^t\mathcal{F}_{,j}\mathcal{F} + {}^t\mathcal{F}\mathcal{F}_{,j} = {}^t\Omega_j\left({}^t\mathcal{F}\mathcal{F}\right) + \left({}^t\mathcal{F}\mathcal{F}\right)\Omega_j$$

$$({}^t\mathcal{F}\mathcal{F})_j = {}^t\Omega_j({}^t\mathcal{F}\mathcal{F}) + ({}^t\mathcal{F}\mathcal{F})\Omega_j$$

<u>linear</u> dift. eq. in ${}^t\mathcal{F}\mathcal{F}$

${}^t\mathcal{F}\mathcal{F}(p_0) =$ initial cond. (Γ^i_{jk})

$$\mathcal{G} := \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow (\mathcal{G})_j = {}^t\Omega_j\mathcal{G} + \mathcal{G}\Omega_j$$

Exercises.

$$\mathcal{G}(p_0) = \begin{pmatrix} g_{ij}(p_0) & 0 \\ 0 & 1 \end{pmatrix}$$

uniqueness of linear system

$$\Rightarrow {}^t\mathcal{F}\mathcal{F} = \mathcal{G}.$$

$$\therefore\ p_{,i} \cdot p_{,j} = g_{ij}$$

$$\nu = a_3$$

$\square$

Exercise 5-1

Problem (Ex. 5-1)

Prove

$$\mathcal{G}_{,j} = {}^t\Omega_j \mathcal{G} + \mathcal{G}\Omega_j$$

Exercise 5-2

Problem (Ex. 5-2)

Let $\theta\colon U \to \mathbb{R}$ be a C^∞-function defined on a simply connected domain U of the uv-plane $\mathbb{R}^2$. Assuming θ satisfies $\theta_{uv} = \sin\theta$, prove that there exists a surface $p\colon U \to \mathbb{R}^3$ whose first and second fundamental forms are

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + dv^2, \qquad II = 2\sin\theta\, du\, dv.$$

Exercise 5-3

Problem (Ex. 5-3)

Let $\sigma\colon U \to \mathbb{R}$ be a C^∞-function defined on a simply connected domain U of the uv-plane $\mathbb{R}^2$. Assuming σ satisfies $\Delta\sigma = -\frac{1}{2}\sinh\sigma$, prove that there exists a surface $p\colon U \to \mathbb{R}^3$ with

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \qquad II = \frac{1}{2}\big((e^{2\sigma}+1)du^2 + (e^{2\sigma}-1)dv^2\big).$$

§6 : Pseudospherical surfaces

$(K = -1)$