# Advanced Topics in Geometry E (MTH.B501)

Pseudospherical surfaces


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## Exercise 5-3

$H = e^{-2\sigma}\frac{L+N}{2}$

$= \frac{1}{2} = \text{const.}$

### Problem (Ex. 5-3)

*Let $\sigma\colon U \to \mathbb{R}$ be a $C^\infty$-function defined on a simply connected domain $U$ of the $uv$-plane $\mathbb{R}^2$. Assuming $\sigma$ satisfies* ✓$\Delta\sigma = -\frac{1}{2}\sinh \cancel{\sigma}\,(2\sigma)$*, prove that there exists a surface $p\colon U \to \mathbb{R}^3$ with*

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \qquad II = \frac{1}{2}\left((e^{2\sigma}+1)du^2 + (e^{2\sigma}-1)dv^2\right).$$

i.e. show that $ds^2$ & II satisfy

$\Leftrightarrow$ Gauss & Codazzi (3 eqs)

"complicated"

the Codazzi eqs hold automatically.

# Exercise 4-2

## Problem (Ex. 4-2)

isothermal parameter

*Assume* $\underline{F = 0}$ *and* $\underline{E = G = e^{2\sigma}}$*, where* $\sigma$ *is a function in* $(u, v)$*. Let* $z = u + iv$ $(i = \sqrt{-1})$ *and define a complex-valued function* $q$ *in* $z$ *by*

$$q(z) := \frac{L(u,v) - N(u,v)}{2} - iM(u,v).$$

$= \frac{1}{2}$

*Prove that the Codazzi equations are equivalent to*

$$\frac{\partial q}{\partial \bar{z}} = e^{2\sigma} \frac{\partial H}{\partial z},$$

*where* $H$ *is the mean curvature, and*

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

# Exercise 5-3

$\mathcal{F}_u = \mathcal{F}\Omega_1 \qquad \mathcal{F}_v = \mathcal{F}\Omega_2$

$$\Omega_1 = \begin{pmatrix} \sigma_u & \sigma_v & -\frac{1}{2}(1+e^{-2\sigma}) \\ -\sigma_v & \sigma_u & 0 \\ \frac{1}{2}(e^{2\sigma}+1) & 0 & 0 \end{pmatrix},$$

$$\Omega_2 = \begin{pmatrix} \sigma_v & -\sigma_u & 0 \\ \sigma_u & \sigma_v & -\frac{1}{2}(1-e^{-2\sigma}) \\ 0 & \frac{1}{2}(e^{2\sigma}-1) & 0 \end{pmatrix}$$

Gauss & Codazzi $\Leftrightarrow (\Omega_1)_v - (\Omega_2)_u - \Omega_1\Omega_2 + \Omega_2\Omega_1 = 0$

$$= \begin{pmatrix} 0 & * & 0 \\ -* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad * = \sigma_{uu} + \sigma_{vv} + \frac{1}{2}\sinh 2\sigma$$

**Backgrounds** construction of CMC surfaces

constant mean curvature

**Hopf's problem**

$\exists$? compact (without boundary) CMC surface in $\mathbb{R}^3$ which is not congruent the round sphere?

$\star$ Hopf : No if the surface $\underset{\text{homeo}}{\cong} S^2$.

自己交叉

~1950 $\star$ Alexandrov : No if the surface has no selfintersection.

1980's • H. Wente, U. Abresch, R. Walter ....

CMC tori:

$$\begin{cases} ds^2 = e^{2\sigma}(du^2 + dv^2) \\ \mathrm{II} = \frac{1}{2}\left((e^{2\sigma}+1)du^2 + (e^{2\sigma}-1)dv^2\right) \end{cases}$$

$\star$ Codazzi are already solved.

compute the period.

Solve $\boxed{\text{Gauss}}$ $\boxed{\Delta\sigma = -\frac{1}{2}\sinh 2\sigma}$ on $\underline{\underline{\mathbb{R}^2}}$

doubly periodic

# Exercise 4-1

$\begin{pmatrix} L & M \\ M & N \end{pmatrix}$

## Problem (Ex. 4-1)

*Assume* $L = N = 0$, *that is,* $II = 2M\,du\,dv = 2h_{12}\,du^1\,du^2$, *Prove that, if the Gaussian curvature* $K$ *is negative constant,*

$$E_v = G_u = 0, \qquad \text{that is, } g_{11,2} = g_{22,1} = 0.$$

Codazzi

Rem

$\theta \not\equiv 0 \pmod{\pi}$

$(0, \pi)$ $\theta \equiv 0 \pmod{\pi} \Rightarrow \det \hat{I} = 0$

## Problem (Ex. 5-2)

*Let $\theta\colon U \to \mathbb{R}$ be a $C^\infty$-function defined on a simply connected domain $U$ of the $uv$-plane $\mathbb{R}^2$. Assuming $\theta$ satisfies $\boxed{\theta_{uv} = \sin\theta}$, prove that there exists a surface $p\colon U \to \mathbb{R}^3$ whose first and second fundamental forms are*

$E_v = G_u = 0 \Leftrightarrow$ Codazzi integrability

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + dv^2, \qquad II = 2\sin\theta\, du\, dv.$$

$$\hat{I} = \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix} \qquad \hat{II} = \begin{pmatrix} 0 & \sin\theta \\ \sin\theta & 0 \end{pmatrix}$$

$$K = \frac{\det \hat{II}}{\det \hat{I}} = \frac{-\sin^2\theta}{1-\cos^2\theta} = -1.$$

# Exercise 3-2

## Problem (Ex. 3-2)

*Assume the first and second fundamental forms of the surface* $p(u^1, u^2)$ *are given in the form*

$$ds^2 = (du^1)^2 + 2\cos\theta\, du^1\, du^2 + (du^2)^2, \qquad II = 2\sin\theta\, du^1\, du^2,$$

*where* $\theta$ *is a smooth function in* $(u^1, u^2)$. *Compute the matrices* $\Omega_j$ $(j = 1, 2)$

# Exercise 2-2

## Problem (Ex. 2-2)

*Let* $\theta = \theta(u, v)$ *be a smooth function on a domain* $U \subset \mathbb{R}^2$ *such that* $0 < \theta < \pi$*, and set*

$$\Omega := \begin{pmatrix} \theta_u \cot\theta & 0 & \cot\theta \\ -\theta_u \csc\theta & 0 & -\csc\theta \\ 0 & \sin\theta & 0 \end{pmatrix}, \Lambda := \begin{pmatrix} 0 & -\theta_v \csc\theta & -\csc\theta \\ 0 & \theta_v \cot\theta & \cot\theta \\ \sin\theta & 0 & 0 \end{pmatrix}.$$

*Prove that the compatibility condition of a system of partial differentail equation*

$$\frac{\partial \mathcal{F}}{\partial u} = \mathcal{F}\Omega, \qquad \frac{\partial \mathcal{F}}{\partial v} = \mathcal{F}\Lambda$$

*is equivalent to*

$$\boxed{\theta_{uv} = \sin\theta.}$$

# Exercise 4-1

## Problem (Ex. 4-1) ✔

*Assume $L = N = 0$, that is, $II = 2M\,du\,dv = 2h_{12}\,du^1\,du^2$, Prove that, if the Gaussian curvature $K$ is negative constant,*

$$E_v = G_u = 0, \qquad \textit{that is,} \qquad g_{11,2} = g_{22,1} = 0.$$

✓ · Pseudospherical surfaces

擬球面 = pseudosphere ($\doteqdot K=-1$)

$\begin{cases} \text{· Beltrami's pseudosphere --} \\ \text{· Dini's pseudosphere --} \end{cases}$