Advanced Topics in Geometry E (MTH.B501)

Pseudospherical surfaces

Kotaro Yamada
kotaro@math.titech.ac.jp
http://www.math.titech.ac.jp/~kotaro/class/2022/geom-e/

Tokyo Institute of Technology

2022/05/31

Exercise 5-3

Problem (Ex. 5-3)

Let $\sigma\colon U \to \mathbb{R}$ be a C^∞-function defined on a simply connected domain U of the uv-plane $\mathbb{R}^2$. Assuming σ satisfies $\Delta\sigma = -\frac{1}{2}\sinh\sigma$, prove that there exists a surface $p\colon U \to \mathbb{R}^3$ with

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \qquad II = \frac{1}{2}\big((e^{2\sigma}+1)du^2 + (e^{2\sigma}-1)dv^2\big).$$

Exercise 4-2

Problem (Ex. 4-2)

Assume $F = 0$ *and* $E = G = e^{2\sigma}$, *where* σ *is a function in* (u, v). *Let* $z = u + iv$ *(*$i = \sqrt{-1}$*) and define a complex-valued function* q *in* z *by*

$$q(z) := \frac{L(u,v) - N(u,v)}{2} - iM(u,v).$$

Prove that the Codazzi equations are equivalent to

$$\frac{\partial q}{\partial \bar{z}} = e^{2\sigma} \frac{\partial H}{\partial z},$$

where H *is the mean curvature, and*

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

Exercise 5-3

$$\Omega_1 = \begin{pmatrix} \sigma_u & \sigma_v & -\frac{1}{2}(1+e^{-2\sigma}) \\ \sigma_v & \sigma_u & 0 \\ \frac{1}{2}(e^{2\sigma}+1) & 0 & \end{pmatrix},$$

$$\Omega_2 = \begin{pmatrix} \sigma_v & -\sigma_u & 0 \\ \sigma_u & \sigma_u & -\frac{1}{2}(1-e^{-2\sigma}) \\ 0 & \frac{1}{2}(e^{2\sigma}-1) & 0 \end{pmatrix}$$

Exercise 4-1

Problem (Ex. 4-1)

Assume $L = N = 0$*, that is,* $II = 2M\,du\,dv = 2h_{12}\,du^1\,du^2$*, Prove that, if the Gaussian curvature* K *is negative constant,*

$$E_v = G_u = 0, \qquad \text{that is,} g_{11,2} = g_{22,1} = 0.$$

Exercise 5-2

Problem (Ex. 5-2)

Let $\theta\colon U \to \mathbb{R}$ be a C^∞-function defined on a simply connected domain U of the uv-plane $\mathbb{R}^2$. Assuming θ satisfies $\theta_{uv} = \sin\theta$, prove that there exists a surface $p\colon U \to \mathbb{R}^3$ whose first and second fundamental forms are

$$ds^2 = du^2 + 2\cos\theta\,du\,dv + dv^2, \qquad II = 2\sin\theta\,du\,dv.$$

Exercise 3-2

Problem (Ex. 3-2)

Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ *are given in the form*

$$ds^2 = (du^1)^2 + 2\cos\theta\, du^1\, du^2 + (du^2)^2, \qquad II = 2\sin\theta\, du^1\, du^2,$$

where θ *is a smooth function in* (u^1, u^2)*. Compute the matrices* Ω_j $(j = 1, 2)$

Exercise 2-2

Problem (Ex. 2-2)

Let $\theta = \theta(u, v)$ be a smooth function on a domain $U \subset \mathbb{R}^2$ such that $0 < \theta < \pi$, and set

$$\Omega := \begin{pmatrix} \theta_u \cot\theta & 0 & \cot\theta \\ -\theta_u \csc\theta & 0 & -\csc\theta \\ 0 & \sin\theta & 0 \end{pmatrix}, \Lambda := \begin{pmatrix} 0 & -\theta_v \csc\theta & -\csc\theta \\ 0 & \theta_v \cot\theta & \cot\theta \\ \sin\theta & 0 & 0 \end{pmatrix}.$$

Prove that the compatibility condition of a system of partial differentail equation

$$\frac{\partial \mathcal{F}}{\partial u} = \mathcal{F}\Omega, \qquad \frac{\partial \mathcal{F}}{\partial v} = \mathcal{F}\Lambda$$

is equivalent to

$$\theta_{uv} = \sin\theta.$$

Exercise 4-1

Problem (Ex. 4-1)

Assume $L = N = 0$, that is, $II = 2M\,du\,dv = 2h_{12}\,du^1\,du^2$, Prove that, if the Gaussian curvature K is negative constant,

$$E_v = G_u = 0, \qquad \text{that is,} \qquad g_{11,2} = g_{22,1} = 0.$$