Advanced Topics in Geometry E (MTH.B501)

construction

Pseudospherical surfaces $K = -1$

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The setting

- $p\colon U \to \mathbb{R}^3$: a regular surface
- $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$: the first fundamental form.
- $II = L\,du^2 + 2M\,du\,dv + N\,dv^2$: the second fundamental form.
- A pseudospherical surface: $K = \frac{LN-M^2}{EG-F^2} = -1$.

$\underline{II = 2M\,du\,dv}$

(u, v): an asymptotic coordinate system

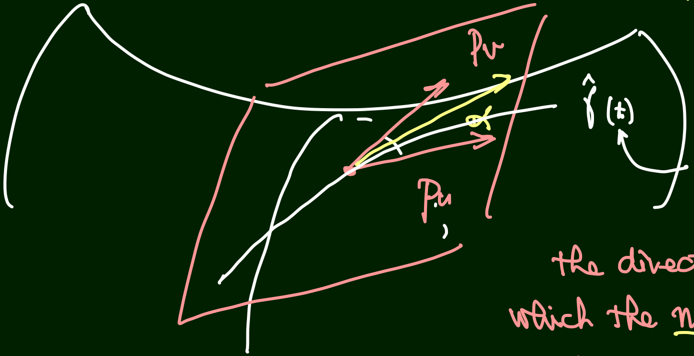
(漸近座標系)

Asymptotic directions （渐近方向）

$\alpha = \alpha p_u + \beta p_v$ is an asymptotic direction

$$\Leftrightarrow \alpha^2 L + 2\alpha\beta M + \beta^2 N = 0.$$

$$(\alpha \ \ \beta)\begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$



$\ddot{\hat{\gamma}} \cdot \nu = \kappa_n$

the direction for which the normal curvature vanishes

法曲率

Asymptotic vector fields 漸近ベクトル場

Lemma

Assume that the Gaussian curvature K is $\boxed{\text{negative}}$ at (u_0, v_0). Then there exists a neighborhood V of (u_0, v_0) and smooth functions α_i, β_i $(i = 1, 2)$ on V such that

$$\boldsymbol{\alpha}_i(u,v) := \alpha_i(u,v)p_u(u,v) + \beta_i(u,v)p_v(u,v) \qquad (i = 1, 2) \quad (1)$$

are two linearly independent asymptotic directions at each $(u, v) \in V$.

Find α, β

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0.$$

$\star$ $\det \hat{\mathrm{I\!I}} < 0 \Rightarrow \exists P : V \xrightarrow{C^\infty} O(2)$

s.t $\quad {}^tP \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} P = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$

C^∞ $\qquad \lambda_1, \lambda_2 > 0$

Erratum

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} := P^{-1} \begin{pmatrix} 1/\sqrt{\lambda_1} \\ 1/\sqrt{\lambda_2} \end{pmatrix}$$

$$\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} := P^{-1} \begin{pmatrix} 1/\sqrt{\lambda_1} \\ -1/\sqrt{\lambda_2} \end{pmatrix}$$

Definition

A parameter (u, v) of the surface $p\colon U \to \mathbb{R}^3$ is called an asymptotic coordinate system or an asymptotic parameter if both the u-curves $u \mapsto p(u, v)$ and the v-curves $v \mapsto p(u, v)$ are asymptotic curves.

p_u p_v : asymptotic v.f.

$= 1p_u + 0p_v$

Lemma

A coordinate system (u, v) of a surface is an asymptotic coordinate system if and only if the second fundamental form is written in the form

$\alpha = 1$ $\beta = 0$

$$✓ \quad II = 2M\,du\,dv,$$

✓ $\alpha^2 L + 2M\alpha\beta + N\beta^2 = 0$

that is, $L = N = 0$. In particular, $M \neq 0$ if the Gaussian curvature does not vanish.

Asymptotic Coordinates

UY: Appendix B-7. Kobayashi-Nomizu (Foundation of Diff. Geom.)

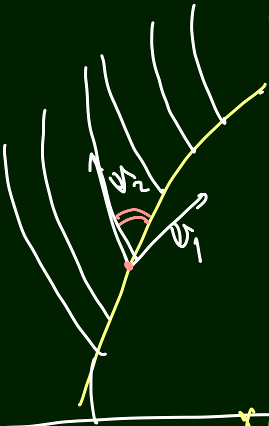
Theorem

Let $p\colon U \to \mathbb{R}^3$ be a regular surface whose Gaussian curvature at (u_0, v_0) is negative. Then there exists a neighborhood V of (u_0, v_0) and a coordinate change $V' \ni (\xi, \eta) \mapsto (u(\xi, \eta), v(\xi, \eta)) \in V$ for which (ξ, η) is an asymptotic coordinate system.

$v_1 = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$ $v_2 = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$

Find a new coordinate system whose coordinate curves tangent to v_j

Outline of the proof



$\gamma_1(s) = (u(s), v(s))$

$\varphi(s,t) = (u(s,t), v(s,t))$

$\frac{\partial \varphi}{\partial t} = v_2(u(s,t), v(s,t))$ ✓

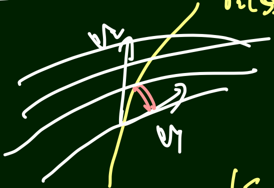
nonlinear ODE if s: fixed.

$(s,t) \mapsto (u,v)$ $\quad \frac{\partial(u,v)}{\partial(s,t)} \neq 0$ at (u_0, v_0)

$\exists\, (u,v) \mapsto (s,t)$

$s(u,v) = \text{const} \Rightarrow$ asympt. curve

$\gamma_2(\xi)$



v_1 - direction $\quad (\xi,\eta) \mapsto (u,v)$

$\exists\, (u,v) \mapsto (\xi,\eta)$

$\xi(u,v) = \text{cnt} \rightarrow$ asympt. curve

(s,ξ): asymptotic coordinates.

Asymptotic Chebyshev net 漸近チェビシェフ網

Theorem

Let $p\colon U \to \mathbb{R}^3$ be a surface with Gaussian curvature -1. Then for each point $(u_0, v_0) \in U$, there exists a neighborhood V and coordinate change $(\xi, \eta) \mapsto (u, v)$ on V such that the first and second fundamental forms are in the form

$$ds^2 = d\xi^2 + 2\cos\theta\, d\xi\, d\eta + d\eta^2, \quad II = 2\sin\theta\, d\xi\, d\eta,$$

where $\theta = \theta(\xi, \eta)$ is a smooth function in (ξ, η) valued in $(0, \pi)$.

$\exists$ asymptotic coordinates

$$\begin{cases} ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \\ II = 2M\,du\,dv \end{cases}$$

$$\begin{cases} ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \\ \mathrm{II} = 2M\,du\,dv \end{cases}$$

$K = -1$: Codazzi $\Leftrightarrow E_v = G_u = 0$

- $E = E(u) > 0$ $\qquad \xi(u) = \int_0^u \sqrt{E}\,du$
- $G = G(v) > 0$ $\qquad \eta(v) = \int_0^v \sqrt{G}\,dv$

$$d\xi^2 = E\,du^2 \qquad d\eta^2 = G\,dv^2$$

$$ds^2 = d\xi^2 + 2\tilde{F}\,d\xi\,d\eta + d\eta^2 \qquad \boxed{1 - \tilde{F}^2 > 0}$$

$$\mathrm{II} = 2\tilde{M}\,d\xi\,d\eta \qquad \tilde{F} = \cos\theta$$

$\tilde{M} = \sin\theta$

(hyperbolic)

$$\theta_{uv} = \sin\theta$$

The Gauss equation for the asymptotic Chebyshev net

$\left(\Delta\sigma = -\frac{1}{2}\sinh 2\sigma\right)$

elliptic

$\theta : \underbrace{U}_{\subset \mathbb{R}^2 \text{ simply connected}} \to (0, \pi) \quad ; \quad \theta_{uv} = \sin\theta$

$\Rightarrow \exists$ pseudospherical surface

To find a simple solution... $\to$ reduce to ODE

$$\theta(u,v) = \theta(u) \Rightarrow \theta_{uv} = 0 \Rightarrow \sin\theta = 0$$

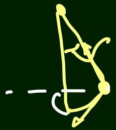
$$\Rightarrow \theta : \text{const.} \qquad \text{(trivial)}$$

Assume $\theta(u,v) = \varphi(u - v)$ $\quad \varphi$: a function of one variable

(To find a simple sol.)

$$\theta_u = \dot{\varphi} \qquad \theta_{uv} = -\ddot{\varphi}$$

$$\boxed{\theta_{uv} = \sin\theta \iff \ddot{\varphi} = -\sin\varphi}$$



(the equation of motion for a pendulum

$$\ddot{\varphi} = -\sin\varphi \qquad \dot{\varphi}\ddot{\varphi} = -\dot{\varphi}\sin\varphi$$

$$\frac{1}{2}\dot{\varphi}^2 = \cos\varphi + \text{const.}$$

$$= 1 - 2\sin^2\frac{\varphi}{2} + \text{const.}$$

$$\left(\frac{\dot{\varphi}}{2}\right)^2 + \sin^2\frac{\varphi}{2} =: e = \text{const} \geq 0 \quad \text{1st integral}$$

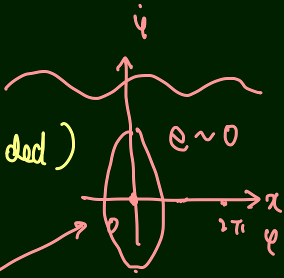
$e = 0 \Rightarrow \varphi$: const.

$e \sim 0 \Rightarrow \varphi$: periodic function

$e \gg 1 \Rightarrow \varphi$: non periodic (unbounded)

$$\boxed{y^2 + \sin^2 x = e}$$

phase



$$\left(\frac{\dot\varphi}{2}\right)^2 + \sin^2\frac{\varphi}{2} = e = 1$$

$$\left(\frac{\dot\varphi}{2}\right)^2 = 1 - \sin^2\frac{\varphi}{2} = \cos^2\frac{\varphi}{2}$$

$$\underline{\underline{\frac{\dot\varphi}{2} = \cos\frac{\varphi}{2}}}$$

$$\boxed{\varphi(0) = 0 \quad \dot\varphi(0) = 2}$$

Exercise 6-1

✓ **Problem (Ex. 6-1)**

Find an explicit solution of (6.5) for $e = 1$, *with initial condition* $\varphi(0) = 0$, $\dot{\varphi}(0) = 2$.

Exercise 6-2

Problem (Ex. 6-2)

For a constant $e \in (0, 1)$, the solution φ of (6.5) with (6.6) is a periodic function. Find the period of such a solution.

elliptic integral