Advanced Topics in Geometry E (MTH.B501)

Pseudospherical surfaces

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The setting

- $p\colon U \to \mathbb{R}^3$: a regular surface
- $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$: the first fundamental form.
- $II = L\,du^2 + 2M\,du\,dv + N\,dv^2$: the second fundamental form.
- A pseudospherical surface: $K = \frac{LN-M^2}{EG-F^2} = -1$.

Asymptotic directions

$\boldsymbol{\alpha} = \alpha p_u + \beta p_v$ is an asymptotic direction

$$\Leftrightarrow \alpha^2 L + 2\alpha\beta M + \beta^2 N = 0.$$

Asymptotic vector fields

Lemma

Assume that the Gaussian curvature K is negative at (u_0, v_0). Then there exists a neighborhoold V of (u_0, v_0) and smooth functions α_i, β_i ($i = 1, 2$) on V such that

$$\boldsymbol{\alpha}_i(u,v) := \alpha_i(u,v)p_u(u,v) + \beta_i(u,v)p_v(u,v) \qquad (i = 1, 2) \quad (1)$$

are two linearly independent asymptotic directions at each $(u, v) \in V$.

Asymptotic Coordinates

Definition

A parameter (u, v) of the surface $p\colon U \to \mathbb{R}^3$ is called an asymptotic coordinat esystem or an asymptotic parameter if both the u-curves $u \mapsto p(u, v)$ and the v-curves $v \mapsto p(u, v)$ are asymptotic curves.

Lemma

A coordinate system (u, v) of a surface is an asymptotic coordinate system if and only if the second fundamental form is written in the form

$$II = 2M\, du\, dv,$$

that is, $L = N = 0$. In particular, $M \neq 0$ if the Gaussian curvature does not vanish.

Asymptotic Coordinates

Theorem

Let $p\colon U \to \mathbb{R}^3$ *be a regular surface whose Gaussian curvature at* (u_0, v_0) *is negative. Then there exists a neighborhood* V *of* (u_0, v_0) *and a coordinate change* $V' \ni (\xi, \eta) \mapsto (u(\xi, \eta), v(\xi, \eta)) \in V$ *for which* (ξ, η) *is an asymptotic coordinate system.*

Asymptotic Chebyshev net

Theorem

Let $p\colon U \to \mathbb{R}^3$ be a surface with Gaussian curvature -1. Then for each point $(u_0, v_0) \in U$, there exists a neighborhood V and coordinate change $(\xi, \eta) \mapsto (u, v)$ on V such that the first and second fundamental forms are in the form

$$ds^2 = d\xi^2 + 2\cos\theta \, d\xi \, d\eta + d\eta^2, \quad II = 2\sin\theta \, d\xi \, d\eta,$$

where $\theta = \theta(\xi, \eta)$ is a smooth function in (ξ, η) valued in $(0, \pi)$.

The sine Gordon equation

$$\theta_{uv} = \sin\theta$$

To find a simple solution...

Exercise 6-1

Problem (Ex. 6-1)

Find an explicit solution of (6.5) for $e = 1$*, with initial condition* $\varphi(0) = 0$*,* $\dot{\varphi}(0) = 2$*.*

Exercise 6-2

Problem (Ex. 6-2)

For a constant $e \in (0, 1)$*, the solution* φ *of (6.5) with (6.6) is a periodic function. Find the period of such a solution.*