Advanced Topics in Geometry E (MTH.B501)

Beltrami's pseudosphere

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2022/06/07

Exercise 6-1

$$\boxed{\ddot{\varphi} = -\sin\varphi}$$

Problem (Ex. 6-1)

Find an explicit solution of (6.5) for $e = 1$, with initial condition $\varphi(0) = 0$, $\dot{\varphi}(0) = 2$.

$$\frac{1}{2}\dot{\varphi}^2 = e + 1 - 2\sin^2\frac{\varphi}{2}$$

$$\varphi(0) = 0 \qquad \dot{\varphi}(0) = 2$$

$$\ddot{F} = -\sin F \qquad \frac{1}{2}\dot{F}^2 = \text{const} + \cos F \qquad \varphi \mapsto F$$

$$= \text{const} + 1 - 2\sin^2\frac{F}{2}$$

$$\cdot\ F(0) = 0 \quad \dot{F}(0) = 2$$

$$2$$

$$\frac{1}{2}\dot{F}^2 = 2\left(1 - \sin^2\frac{F}{2}\right) = 2\cos^2\frac{F}{2} \qquad \left(\frac{\dot{F}}{2}\right)^2 = \cos^2\frac{F}{2}$$

$$\frac{\dot{F}}{2} = \cos\frac{F}{2} \quad ; \quad 1 = \frac{\dot{F}/2}{\cos F/2}$$

$$t = \int_0^t du = \int_0^t \frac{\dot{F}(u)/2}{\cos F(u)/2}\, du = \int_0^{F(t)/2} \frac{dv}{\cos v}$$

$$= \int_0^{\frac{F}{2}} \frac{\cos v}{1 - \sin^2 v}\, dv = \int_0^{\frac{F}{2}} \frac{d(\sin v)}{1 - \sin^2 v}$$

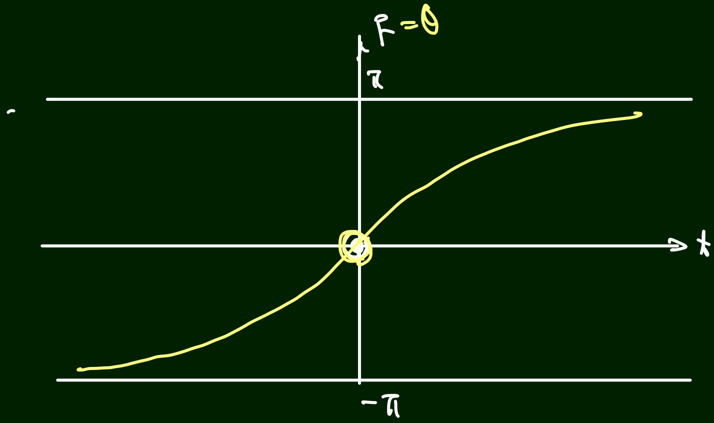
$$t = \int_0^{F(t)/2} \frac{d\sin u}{1-\sin^2 u} = \frac{1}{2}\log \frac{1+\sin u}{1-\sin u}\Big|_0^{F/2}$$

$$= \frac{1}{2}\log \frac{1+\sin F/2}{1-\sin F/2} = \frac{1}{2}\log \frac{(\cos F/4 + \sin F/4)^2}{(\cos F/4 - \sin F/4)^2}$$

$$= \log \frac{\cos F/4 + \sin F/4}{\cos F/4 - \sin F/4} = \log \frac{\cancel{\sqrt{2}}\cos\left(\frac{F}{4}-\frac{\pi}{4}\right)}{\cancel{\sqrt{2}}\cos\left(\frac{F}{4}+\frac{\pi}{4}\right)}$$

$$= \log \frac{\sin\left(\frac{F}{4}+\frac{\pi}{4}\right)}{\cos\left(\frac{F}{4}+\frac{\pi}{4}\right)} = \log \tan\left(\frac{F}{4}+\frac{\pi}{4}\right)$$

$$\boxed{F(t) = 4\tan^{-1} e^t - \pi}$$



F = θ
π
x
−π

Exercise 6-2

Problem (Ex. 6-2)

For a constant $e \in (0,1)$, the solution φ of (6.5) with (6.6) is a periodic function. Find the period of such a solution.

$$\begin{pmatrix} F(t) & \dot{F}(t) \end{pmatrix} =: \begin{pmatrix} x(t) & y(t) \end{pmatrix}$$

$$\frac{y^2}{2} = e+1 - \cancel{2}\sin^2\frac{x}{2}$$

$$\left(\frac{dx}{dt}\right)^2 = 2(e+1) - 4\sin^2\frac{x}{2}$$

$$\frac{dx}{dt} = \sqrt{2(e+1) - 4\sin^2\frac{x}{2}}$$

$$\frac{dt}{dx} = \frac{1}{\sqrt{2(e+1) - 2\omega^2 \frac{x}{2}}}$$

period

$$\frac{T}{4} = \int_{x=0}^{x=*} \frac{dx}{\sqrt{}}$$

elliptic integral.