Advanced Topics in Geometry E (MTH.B501)

Beltrami's pseudosphere

Kotaro Yamada
kotaro@math.titech.ac.jp
http://www.math.titech.ac.jp/~kotaro/class/2022/geom-e/

Tokyo Institute of Technology

2022/06/07

Beltrami's pseudosphere (data)

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + dv^2$$

$$II = 2\sin\theta\, du\, dv$$

$\theta_{uv} = \sin\theta$

$$\theta(u,v) = F(u-v)$$ ($\ddot{F} = -\sin F$)

$$F(t) = 4\tan^{-1} e^t - \pi$$ a special sol.

$$\cos F(t) = 1 - 2\tanh^2 t$$

$$\sin F(t) = 2\,\mathrm{sech}\, t \tanh t$$

Rem $\hat{I} = \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}$ $\det \hat{I} = 1 - \cos^2\theta = \sin^2\theta$

$\theta \equiv 0 \pmod{\pi}$: singularity

$t = 0$: $(u - v = 0)$: singularity

$$F(t) = 4\tan^{-1} e^{t} - \pi$$

$$\cos F = -\cos\left(4\tan^{-1} e^{t}\right) = +1 - 2\cos^2\left(2\tan^{-1} e^{t}\right)$$

$$= +1 - 2\left(\frac{1-e^{2t}}{1+e^{2t}}\right)^2 = 1 - 2\left(\frac{e^{t}-e^{-t}}{e^{t}+e^{-t}}\right)^2$$

$$= 1 - 2\tanh^2 t$$

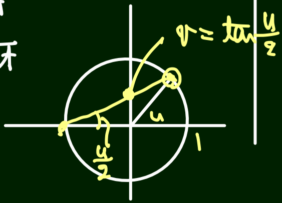
$$\sin F = -\sin\left(4\tan^{-1} e^{t}\right)$$

$$= -2\sin\left(2\tan^{-1} e^{t}\right)\cdot\cos\left(2\cdots\right)$$

$$= -2\,\frac{2e^{t}}{1+e^{2t}}\cdot\frac{1-e^{2t}}{1+e^{2t}}$$

$$= 2\,\frac{2}{e^{t}+e^{-t}}\,\frac{e^{t}-e^{-t}}{e^{t}+e^{-t}}$$

$$= 2\,\mathrm{sech}\,t\,\tanh t$$



$$\tan\frac{u}{2} = v$$

- $\cos u = \dfrac{1-v^2}{1+v^2}$
- $\sin u = \dfrac{2v}{1+v^2}$

stereographic projection

Beltrami's pseudosphere (parameter change)

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + dv^2$$

$$II = 2\sin\theta\, du\, dv$$

$$\theta(u,v) = F(u-v)$$

$$F(\xi) = 4\tan^{-1} e^{\xi} - \pi$$

$$\cos F(\xi) = 1 - 2\tanh^2 \xi$$

$$\sin F(\xi) = 2\,\mathrm{sech}\, t \tanh t$$

$\xi \qquad \xi$

$$(\xi,\eta) = (u-v, u+v), \qquad (u,v) = \left(\frac{1}{2}(\xi+\eta), \frac{1}{2}(-\xi,\eta)\right)$$

The Gauss-Weingarten forula

$$ds^2 = \tanh^2\xi\, d\xi^2 + \operatorname{sech}^2\xi\, d\eta^2 \qquad II = -\tanh\xi \operatorname{sech}\xi (d\xi^2 - d\eta^2)$$

$$\begin{aligned}
p_{\xi\xi} &= \operatorname{sech}\xi(\operatorname{csch}\xi\, p_\xi - \tanh\xi\, \nu), & \nu_\xi &= \operatorname{csch}\xi\, p_\xi,\\
p_{\xi\eta} &= -\tanh\xi\, p_\eta, & \nu_\eta &= -\sinh\xi\, p_\eta.\\
p_{\eta\eta} &= \operatorname{sech}\xi(\operatorname{csch}\xi\, p_\xi + \tanh\xi\, \nu), & &
\end{aligned}$$

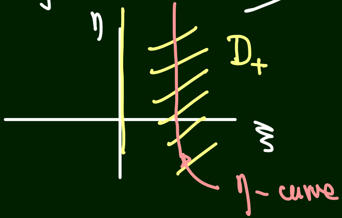
Gauss - Weingarten Formula

$\exists$ surface $p: D_+ \to \mathbb{R}^3$

$\Uparrow$

(The fundamental theorem

sine Gordon



$(\xi > 0)$

The η-curves ξ: fixed

$$\sigma = \sigma^{\xi}(\eta) = p(\xi, \eta) \qquad |p_\eta|^2 = \operatorname{sech}^2 \xi = \text{const}$$

$$\sigma' = p_\eta = (\operatorname{sech} \xi) \cdot e \qquad |e| = 1.$$

$s = \eta \operatorname{sech} \xi$: the arclength

$$\frac{d\sigma}{ds} = e = \cosh \xi \, p_\eta$$

$$\frac{d^2\sigma}{ds^2} = \frac{d\eta}{ds}\frac{de}{d\eta} = \cosh \xi \frac{de}{d\eta} = \cosh^2 \xi \, p_{\eta\eta}$$

$$= \cosh^2 \xi \operatorname{sech} \xi \,(\underbrace{\operatorname{csch} \xi \, p_\xi + \tanh \xi \, \nu}_{= n})$$

$\parallel \sinh \xi$

$$= \cosh \xi \; n$$

$$(n = \operatorname{csch} p_\xi + \tanh \xi \, \nu \qquad |p_\xi|^2 = \tanh^2 \xi \qquad |n| = 1$$

the curvature constant. (the principal normal)

$\mathbb{b}$ = the binormal of $\sigma = \sigma_\xi$ $\qquad$ $\mathbb{b} = \mathbb{t} \times \mathbb{n}$

$$\mathbb{b} = (\cosh\xi \cdot p_\eta) \times (\operatorname{csch}\xi\, p_\xi + \tanh\xi\, \nu)$$

$$= \cosh\xi\Big(-\operatorname{csch}\xi \tanh\xi \operatorname{sech}\xi\, \nu$$

$$+ \tanh\xi \operatorname{csch}\xi\, p_\xi\Big)$$

$\because p_\xi \times p_\eta = \tanh\xi \operatorname{sech}\xi\, \nu$

$\because p_\eta \times \nu = \operatorname{csch} p_\xi$

$$= \cosh\xi\left(-\operatorname{sech}^2\xi\, \nu + \operatorname{sech}\xi\, p_\xi\right)$$

$$(\underbrace{p_\xi, p_\eta, \nu}_{\text{orthogonal}})$$

$$= p_\xi - \operatorname{sech}\xi\, \nu$$: <u>binormal of σ</u>

$$\frac{\operatorname{sech}\xi}{\tanh\xi}$$

$\mathbb{b} = p_\xi - \operatorname{sech}\xi\, \nu.$ (binormal)

<u>Claim</u> $\mathbb{b}_\xi = \mathbb{b}_\eta = 0$ $\therefore$ $\mathbb{b}$: const, unit.

$\mathbb{b}_\eta = 0$ ($\mathbb{b}$: const along σ) $\Rightarrow$

(the image of σ lies on a plane perpendicular to $\mathbb{b}$)

- the curvature of $\sigma = \sigma^\xi$: $\cosh\xi$
- arclength $s = \eta \operatorname{sech}\xi$

σ: a circle of radius $\operatorname{sech}\xi$ (length: $2\pi \operatorname{sech}\xi$)

$\Rightarrow p(\xi, \eta + 2\pi) = p(\xi, \eta)$

"The surface is foliated by circles in parallel planes"

The center of the circle $\sigma^{\xi}(\eta) = p(\xi, \eta)$

$$\mathbb{C}(\xi) = p(\xi, \eta) + \frac{1}{\kappa(\xi)} n(\xi, \eta)$$

$$= p + \operatorname{sech}\xi \left(\operatorname{csch}\xi \, p_{\xi} + \tanh\xi \, \nu\right)$$

By Gauss & Weingarten:

$$= p_{\xi} - \operatorname{sech}\xi \, \nu$$

$$\mathbb{C}_{\xi} = \cdots = \tanh^2\xi \, \mathbb{b}$$

$$\mathbb{C}_{\eta} = 0$$

$$\begin{cases} \boxed{\mathbb{b} : \text{constant, unit}} \leftarrow \text{1st integral.} \\ \gamma^{\xi} = p(\xi, \cdot) : \text{a circle of radius } \operatorname{sech}\xi \text{ on } \mathbb{b}^{\perp} \\ \qquad \text{centered at } \mathbb{c} \qquad \mathbb{c}_{\xi} = (\tanh^2\xi)\,\mathbb{b} \\ p(\xi, \eta + 2\pi) = p(\xi, \eta) \end{cases}$$

By a rotation in $\mathbb{R}^3$, we can set $\mathbb{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\mathbb{c}_{\xi} = \begin{pmatrix} 0 \\ 0 \\ \tanh^2\xi \end{pmatrix} \qquad \mathbb{c} = \begin{pmatrix} 0 \\ 0 \\ \xi - \tanh^2\xi \end{pmatrix}$$

$$\tanh^2\xi = 1 - \operatorname{sech}^2\xi$$

$$p = \begin{pmatrix} \operatorname{sech}\xi \cos\eta \\ \operatorname{sech}\xi \sin\eta \\ \xi - \tanh^2\xi \end{pmatrix}$$

Beltrami's pseudosphere.

$$p(\xi,\eta) = {}^{t}(\operatorname{sech}\xi\cos\eta, \operatorname{sech}\xi\,\xi\eta, \xi - \tanh\xi)$$

cos

Rem $\quad \theta(u,v) = F(u-v) \quad F = \cdots$ Beltrami's p.s.

$$\theta(u,v) = F\left(au - \frac{1}{a}v\right) \quad (a \neq 0, \text{const})$$

$$\theta_{uv} = -\ddot{F} = \sin F = \sin\theta$$

$\Rightarrow \exists$ surface $\left(\overset{\xi}{au - \frac{1}{a}v} > 0\right)$

(a one parameter deformation of the pseudo sphere)

$$\eta = au + \frac{1}{a}v$$

η curve : $\kappa = \text{const}$, τ : non zero const

a helix

Dini's pseudosphere.

✓ Rem $\quad \exists$ Bäcklund transf : ($K = -1$ surface $\longrightarrow K = -1$ surface)