Advanced Topics in Geometry E (MTH.B501)

Beltrami's pseudosphere

Kotaro Yamada
kotaro@math.titech.ac.jp
http://www.math.titech.ac.jp/~kotaro/class/2022/geom-e/

Tokyo Institute of Technology

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Beltrami's pseudosphere (data)

$$
\begin{aligned}
ds^2 &= du^2 + 2\cos\theta \, du \, dv + dv^2 \\
II &= 2\sin\theta \, du \, dv \\
\theta(u,v) &= F(u-v) \\
F(t) &= 4\tan^{-1} e^t - \pi \\
\cos F(t) &= 1 - 2\tanh^2 t \\
\sin F(t) &= 2\,\mathrm{sech}\, t \tanh t
\end{aligned}
$$

Beltrami's pseudosphere (parameter change)

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + dv^2$$
$$II = 2\sin\theta\, du\, dv$$
$$\theta(u,v) = F(u-v)$$
$$F(\xi) = 4\tan^{-1} e^{\xi} - \pi$$
$$\cos F(\xi) = 1 - 2\tanh^2 \xi$$
$$\sin F(\xi) = 2\,\mathrm{sech}\, t \tanh t$$

$$(\xi, \eta) = (u - v, u + v), \qquad (u, v) = \left(\frac{1}{2}(\xi + \eta), \frac{1}{2}(-\xi, \eta)\right)$$

The Gauss-Weingarten forula

$$ds^2 = \tanh^2\xi\, d\xi^2 + \operatorname{sech}^2\xi\, d\eta^2 II \quad = -\tanh\xi \operatorname{sech}\xi (d\xi^2 - d\eta^2)$$

$$\begin{aligned}
p_{\xi\xi} &= \operatorname{sech}\xi(\operatorname{csch}\xi\, p_\xi - \tanh\xi\,\nu), & \nu_\xi &= \operatorname{csch}\xi\, p_\xi, \\
p_{\xi\eta} &= -\tanh\xi\, p_\eta, & \nu_\eta &= -\sinh\xi\, p_\eta. \\
p_{\eta\eta} &= \operatorname{sech}\xi(\operatorname{csch}\xi\, p_\xi + \tanh\xi\,\nu),
\end{aligned}$$

The η-curves

Beltrami's pseudosphere

$$p(\xi, \eta) = {}^t(\operatorname{sech}\xi \cos\eta, \operatorname{sech}\xi \xi\eta, \xi - \tanh\xi)$$