1 Linear Ordinary Differential Equations

The fundamental theorem for ordinary differential equations. Consider a function

$$\boldsymbol{f}\colon I \times U \ni (t, \boldsymbol{x}) \longmapsto \boldsymbol{f}(t, \boldsymbol{x}) \in \mathbb{R}^m \tag{1.1}$$

of class C^1, where $I \subset \mathbb{R}$ is an interval and $U \subset \mathbb{R}^m$ is a domain in the Euclidean space $\mathbb{R}^m$. For any fixed $t_0 \in I$ and $\boldsymbol{x}_0 \in U$, the condition

$$\frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{f}\bigl(t, \boldsymbol{x}(t)\bigr), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{1.2}$$

of an $\mathbb{R}^m$-valued function $t \mapsto \boldsymbol{x}(t)$ is called the *initial value problem of ordinary differential equation* for unknown function $\boldsymbol{x}(t)$. A function $\boldsymbol{x}\colon I \to U$ satisfying (1.2) is called a *solution* of the initial value problem.

Fact 1.1 (The existence theorem for ODE's)**.** *Let $\boldsymbol{f}\colon I \times U \to \mathbb{R}^m$ be a C^1-function as in* (1.1)*. Then, for any $\boldsymbol{x}_0 \in U$ and $t_0 \in I$, there exists a positive number ε and a C^1-function $\boldsymbol{x}\colon I \cap (t_0 - \varepsilon, t_0 + \varepsilon) \to U$ satisfying* (1.2)*.*

Consider two solutions $\boldsymbol{x}_j\colon J_j \to U$ ($j = 1, 2$) of (1.2) defined on subintervals $J_j \subset I$ containing t_0. Then the function $\boldsymbol{x}_2$ is said to be an *extension* of $\boldsymbol{x}_1$ if $J_1 \subset J_2$ and $\boldsymbol{x}_2|_{J_1} = \boldsymbol{x}_1$. A solution $\boldsymbol{x}$ of (1.2) is said to be *maximal* if there are no non-trivial extension of it.

Fact 1.2 (The uniqueness for ODE's)**.** *The maximal solution of* (1.2) *is unique.*

Fact 1.3 (Smoothness of the solutoins)**.** *If $\boldsymbol{f}\colon I \times U \to \mathbb{R}^m$ is of class C^r ($r = 1, \dots, \infty$), the solution of* (1.2) *is of class C^{r+1}. Here, $\infty + 1 = \infty$, as a convention.*

Let $V \subset \mathbb{R}^k$ be another domain of $\mathbb{R}^k$ and consider a C^∞-function

$$\boldsymbol{h}\colon I \times U \times V \ni (t, \boldsymbol{x}; \boldsymbol{\alpha}) \mapsto \boldsymbol{h}(t, \boldsymbol{x}; \boldsymbol{\alpha}) \in \mathbb{R}^m. \tag{1.3}$$

For fixed $t_0 \in I$, we denote by $\boldsymbol{x}(t; \boldsymbol{x}_0, \boldsymbol{\alpha})$ the (unique, maximal) solution of (1.2) for $\boldsymbol{f}(t, \boldsymbol{x}) = \boldsymbol{h}(t, \boldsymbol{x}; \boldsymbol{\alpha})$. Then

Fact 1.4. *The map $(t, \boldsymbol{x}_0; \boldsymbol{\alpha}) \mapsto \boldsymbol{x}(t; \boldsymbol{x}_0, \boldsymbol{\alpha})$ is of class C^∞.*

Example 1.5. (1) Let $m = 1$, $I = \mathbb{R}$, $U = \mathbb{R}$ and $f(t, x) = \lambda x$, where λ is a constant. Then $x(t) = x_0 \exp(\lambda t)$ defined on $\mathbb{R}$ is the maximal solution to

$$\frac{d}{dt}x(t) = f(t, x(t)) = \lambda x(t), \qquad x(0) = x_0.$$

(2) Let $m = 2$, $I = \mathbb{R}$, $U = \mathbb{R}^2$ and $\boldsymbol{f}(t; (x, y)) = (y, -\omega^2 x)$, where ω is a constant. Then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \cos\omega t + \frac{y_0}{\omega}\sin\omega t \\ -x_0\omega\sin\omega t + y_0\cos\omega t \end{pmatrix}$$

is the unique solution of

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ -\omega^2 x(t) \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

defined on $\mathbb{R}$. This differential equation can be considered a single equation

$$\frac{d^2}{dt^2}x(t) = -\omega^2 x(t), \quad x(0) = x_0, \quad \frac{dx}{dt}(0) = y_0$$

of order 2.

(3) Let $m = 1$, $I = \mathbb{R}$, $U = \mathbb{R}$ and $f(t, x) = 1 + x^2$. Then $x(t) = \tan t$ defined on $(-\frac{\pi}{2}, \frac{\pi}{2})$ is the unique maximal solution of the initial value problem

$$\frac{dx}{dt} = 1 + x^2, \qquad x(0) = 0.$$

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Linear Ordinary Differential Equations. The ordinary differential equation (1.2) is said to be *linear* if the function (1.1) is a linear function in $\boldsymbol{x}$, that is, a linear differential equation is in a form

$$\frac{d}{dt}\boldsymbol{x}(t) = A(t)\boldsymbol{x}(t) + \boldsymbol{b}(t),$$

where $A(t)$ and $\boldsymbol{b}(t)$ are $m \times m$-matrix-valued and $\mathbb{R}^m$-valued functions in t.

For the sake of later use, we consider, in this lecture, the special form of linear differential equation for matrix-valued unknown functions as follows: Let $\mathrm{M}_n(\mathbb{R})$ be the set of $n \times n$-matrices with real components, and take functions

$$\Omega\colon I \longrightarrow \mathrm{M}_n(\mathbb{R}), \quad \text{and} B\colon I \longrightarrow \mathrm{M}_n(\mathbb{R}),$$

where $I \subset \mathbb{R}$ is an interval. Identifying $\mathrm{M}_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$, we assume Ω and B are continuous functions (with respect to the topology of $\mathbb{R}^{n^2} = \mathrm{M}_n(\mathbb{R})$). Then we can consider the linear ordinary differential equation for matrix-valued unknown $X(t)$ as

$$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad X(t_0) = X_0, \tag{1.4}$$

where X_0 is given constant matrix.

Then, the fundamental theorem of *linear* ordinary equation states that *the maximal solution of* (1.4) *is defined on whole* I. To prove this, we prepare some materials related to matrix-valued functions.

Preliminaries: Matrix Norms. Denote by $\mathrm{M}_n(\mathbb{R})$ the set of $n \times n$-matrices with real components, which can be identified the vector space $\mathbb{R}^{n^2}$. In particular, the Euclidean norm of $\mathbb{R}^{n^2}$ induces a norm

$$|X|_{\mathrm{E}} = \sqrt{\operatorname{tr}({}^tXX)} = \sqrt{\sum_{i,j=1}^{n} x_{ij}^2} \tag{1.5}$$

on $\mathrm{M}_n(\mathbb{R})$. On the other hand, we let

$$|X|_{\mathrm{M}} := \sup\left\{\frac{|X\boldsymbol{v}|}{|\boldsymbol{v}|}\,;\, \boldsymbol{v} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}\right\}, \tag{1.6}$$

where $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^n$.

Lemma 1.6. (1) *The map* $X \mapsto |X|_{\mathrm{M}}$ *is a norm of* $\mathrm{M}_n(\mathbb{R})$.

(2) *For* X, $Y \in \mathrm{M}_n(\mathbb{R})$, *it holds that* $|XY|_{\mathrm{M}} \leqq |X|_{\mathrm{M}}\,|Y|_{\mathrm{M}}$.

(3) *Let* $\lambda = \lambda(X)$ *be the maximum eigenvalue of semi-positive definite symmetric matrix* ${}^tXX$. *Then* $|X|_{\mathrm{M}} = \sqrt{\lambda}$ *holds.*

(4) $(1/\sqrt{n})|X|_{\mathrm{E}} \leqq |X|_{\mathrm{M}} \leqq |X|_{\mathrm{E}}$.

(5) *The map* $|\cdot|_{\mathrm{M}}\colon \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}$ *is continuous with respect to the Euclidean norm.*

Proof. Since $|X\boldsymbol{v}|/|\boldsymbol{v}|$ is invariant under scalar multiplications to $\boldsymbol{v}$, we have $|X|_{\mathrm{M}} = \sup\{|X\boldsymbol{v}|\,;\, \boldsymbol{v} \in S^{n-1}\}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Since $S^{n-1} \ni \boldsymbol{x} \mapsto |A\boldsymbol{x}| \in \mathbb{R}$ is a continuous function defined on a compact space, it takes the maximum. Thus, the right-hand side of (1.6) is well-defined. It is easy to verify that $|\cdot|_{\mathrm{M}}$ satisfies the axiom of the norm[1].

[1] $|X|_{\mathrm{M}} > 0$ whenever $X \neq O$, $|\alpha X|_{\mathrm{M}} = |\alpha|\,|X|_{\mathrm{M}}$, and the triangle inequality.

Since $A := {}^tXX$ is positive semi-definite, the eigenvalues λ_j ($j = 1,\dots,n$) are non-negative real numbers. In particular, there exists an orthonormal basis $[\boldsymbol{a}_j]$ of $\mathbb{R}^n$ satisfying $A\boldsymbol{a}_j = \lambda_j\boldsymbol{a}_j$ ($j = 1,\dots,n$). Let λ be the maximum eigenvalue of A, and write $\boldsymbol{v} = v_1\boldsymbol{a}_1 + \cdots + v_n\boldsymbol{a}_n$. Then it holds that

$$\langle X\boldsymbol{v}, X\boldsymbol{v}\rangle = \lambda_1 v_1^2 + \cdots + \lambda_n v_n^2 \leqq \lambda \langle \boldsymbol{v}, \boldsymbol{v}\rangle\,,$$

where $\langle\ ,\ \rangle$ is the Euclidean inner product of $\mathbb{R}^n$. The equality of this inequality holds if and only if $\boldsymbol{v}$ is the λ-eigenvector, proving (3). Noticing the norm (3.2) is invariant under conjugations $X \mapsto {}^tPXP$ ($P \in \mathrm{O}(n)$), we obtain $|X|_{\mathrm{E}} = \sqrt{\lambda_1^2 + \cdots + \lambda_n^2}$ by diagonalizing ${}^tXX$ by an orthogonal matrix P. Then we obtain (4). Hence two norms $|\cdot|_{\mathrm{E}}$ and $|\cdot|_{\mathrm{M}}$ induce the same topology as $\mathrm{M}_n(\mathbb{R})$. In particular, we have (5). □

Preliminaries: Matrix-valued Functions.

Lemma 1.7. *Let X and Y be C^∞-maps defined on a domain $U \subset \mathbb{R}^m$ into $\mathrm{M}_n(\mathbb{R})$. Then*

(1) $\dfrac{\partial}{\partial u_j}(XY) = \dfrac{\partial X}{\partial u_j}Y + X\dfrac{\partial Y}{\partial u_j}$,

(2) $\dfrac{\partial}{\partial u_j}\det X = \operatorname{tr}\left(\widetilde{X}\dfrac{\partial X}{\partial u_j}\right)$, *and*

(3) $\dfrac{\partial}{\partial u_j}X^{-1} = -X^{-1}\dfrac{\partial X}{\partial u_j}X^{-1}$,

where $\widetilde{X}$ is the cofactor matrix of X, and we assume in (3) *that X is a regular matrix.*

Proof. The formula (1) holds because the definition of matrix multiplication and the Leibnitz rule, Denoting $' = \partial/\partial u_j$,

$$O = (\mathrm{id})' = (X^{-1}X)' = (X^{-1})X' + (X^{-1})'X$$

implies (3), where id is the identity matrix.

Decompose the matrix X into column vectors as $X = (\boldsymbol{x}_1,\dots,\boldsymbol{x}_n)$. Since the determinant is multi-linear form for n-tuple of column vectors, it holds that

$$(\det X)' = \det(\boldsymbol{x}_1', \boldsymbol{x}_2, \dots, \boldsymbol{x}_n) + \det(\boldsymbol{x}_1, \boldsymbol{x}_2', \dots, \boldsymbol{x}_n) + \cdots + \det(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n').$$

Then by cofactor expansion of the right-hand side, we obtain (2). □

Proposition 1.8. *Assume two C^∞ matrix-valued functions $X(t)$ and $\Omega(t)$ satisfy*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0. \tag{1.7}$$

Then

$$\det X(t) = (\det X_0)\exp\int_{t_0}^{t}\operatorname{tr}\Omega(\tau)\,d\tau \tag{1.8}$$

holds. In particular, if $X_0 \in \mathrm{GL}(n,\mathbb{R})$,[2] then $X(t) \in \mathrm{GL}(n,\mathbb{R})$ for all t.

Proof. By (2) of Lemma 1.7, we have

$$\begin{aligned}\frac{d}{dt}\det X(t) &= \operatorname{tr}\left(\widetilde{X}(t)\frac{dX(t)}{dt}\right) = \operatorname{tr}\left(\widetilde{X}(t)X(t)\Omega(t)\right)\\ &= \operatorname{tr}\bigl(\det X(t)\Omega(t)\bigr) = \det X(t)\operatorname{tr}\Omega(t).\end{aligned}$$

Here, we used the relation $\widetilde{X}X = X\widetilde{X} = (\det X)\,\mathrm{id}$[3]. Hence $\frac{d}{dt}\bigl(\rho(t)^{-1}\det X(t)\bigr) = 0$, where $\rho(t)$ is the right-hand side of (1.8). □

[2] $\mathrm{GL}(n,\mathbb{R}) = \{A \in \mathrm{M}_n(\mathbb{R})\,;\,\det A \neq 0\}$: the general linear group.

[3] In this lecture, id denotes the identity matrix.

Corollary 1.9. *If $\Omega(t)$ in (1.7) satisfies $\operatorname{tr}\Omega(t) = 0$, $\det X(t)$ is constant. In particular, if $X_0 \in \mathrm{SL}(n,\mathbb{R})$, X is a function valued in $\mathrm{SL}(n,\mathbb{R})$* [4].

Proposition 1.10. *Assume $\Omega(t)$ in (1.7) is skew-symmetric for all t, that is, ${}^t\Omega + \Omega$ is identically O. If $X_0 \in \mathrm{O}(n)$ (resp. $X_0 \in \mathrm{SO}(n)$)*[5]*, then $X(t) \in \mathrm{O}(n)$ (resp. $X(t) \in \mathrm{SO}(n)$) for all t.*

Proof. By (1) in Lemma 1.7,

$$\begin{aligned}\frac{d}{dt}(X\,{}^tX) &= \frac{dX}{dt}\,{}^tX + X\,{}^t\!\left(\frac{dX}{dt}\right)\\ &= X\Omega\,{}^tX + X\,{}^t\Omega\,{}^tX = X(\Omega + {}^t\Omega)\,{}^tX = O.\end{aligned}$$

Hence $X\,{}^tX$ is constant, that is, if $X_0 \in \mathrm{O}(n)$,

$$X(t)\,{}^tX(t) = X(t_0)\,{}^tX(t_0) = X_0\,{}^tX_0 = \mathrm{id}\,.$$

If $X_0 \in \mathrm{O}(n)$, this proves the first case of the proposition. Since $\det A = \pm 1$ when $A \in \mathrm{O}(n)$, the second case follows by continuity of $\det X(t)$. □

Preliminaries: Norms of Matrix-Valued functions. Let $I = [a,b]$ be a closed interval, and denote by $C^0(I, \mathrm{M}_n(\mathbb{R}))$ the set of continuous functions $X\colon I \to \mathrm{M}_n(\mathbb{R})$. For any positive number k, we define

$$||X||_{I,k} := \sup\left\{e^{-kt}|X(t)|_{\mathrm{M}}\,;\, t \in I\right\} \tag{1.9}$$

for $X \in C^0(I, \mathrm{M}_n(\mathbb{R}))$. When $k = 0$, $||\cdot||_{I,0}$ is the *uniform norm* for continuous functions, which is complete. Similarly, one can prove the following in the same way:

Lemma 1.11. *The norm $||\cdot||_{I,k}$ on $C^0(I, \mathrm{M}_n(\mathbb{R}))$ is complete.*

Linear Ordinary Differential Equations. We prove the fundamental theorem for *linear* ordinary differential equations.

Proposition 1.12. *Let $\Omega(t)$ be a C^∞-function valued in $\mathrm{M}_n(\mathbb{R})$ defined on an interval I. Then for each $t_0 \in I$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0,\mathrm{id}}(t)$ such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = \mathrm{id}\,. \tag{1.10}$$

Proof. Uniqueness: Assume $X(t)$ and $Y(t)$ satisfy (1.10). Then

$$Y(t) - X(t) = \int_{t_0}^{t}\bigl(Y'(\tau) - X'(\tau)\bigr)\,d\tau = \int_{t_0}^{t}\bigl(Y(\tau) - X(\tau)\bigr)\Omega(\tau)\,d\tau \qquad \left({}' = \frac{d}{dt}\right)$$

holds. Hence for an arbitrary closed interval $J \subset I$,

$$\begin{aligned}|Y(t) - X(t)|_{\mathrm{M}} &\leqq \left|\int_{t_0}^{t}\left|\bigl(Y(\tau) - X(\tau)\bigr)\Omega(\tau)\right|_{\mathrm{M}} d\tau\right| \leqq \left|\int_{t_0}^{t}|Y(\tau) - X(\tau)|_{\mathrm{M}}\,|\Omega(\tau)|_{\mathrm{M}}\,d\tau\right|\\ &= \left|\int_{t_0}^{t} e^{-k\tau}\,|Y(\tau) - X(\tau)|_{\mathrm{M}}\,e^{k\tau}\,|\Omega(\tau)|_{\mathrm{M}}\,d\tau\right| \leqq ||Y - X||_{J,k}\sup_J|\Omega|_{\mathrm{M}}\left|\int_{t_0}^{t} e^{k\tau}\,d\tau\right|\\ &= ||Y - X||_{J,k}\frac{\sup_J|\Omega|_{\mathrm{M}}}{|k|}e^{kt}\left|1 - e^{-k(t-t_0)}\right| \leqq ||Y - X||_{J,k}\sup_J|\Omega|_{\mathrm{M}}\frac{e^{kt}}{|k|}\end{aligned}$$

[4] $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{M}_n(\mathbb{R})\,;\, \det A = 1\}$; the special lienar group.

[5] $\mathrm{O}(n) = \{A \in \mathrm{M}_n(\mathbb{R})\,;\, {}^tAA = A\,{}^tA = \mathrm{id}\}$: the orthogonal group; $\mathrm{SO}(n) = \{A \in \mathrm{O}(n)\,;\, \det A = 1\}$: the special orthogonal group.

holds for $t \in J$. Thus, for an appropriate choice of $k \in \mathbb{R}$, it holds that

$$||Y - X||_{J,k} \leqq \frac{1}{2}||Y - X||_{J,k},$$

that is, $||Y - X||_{J,k} = 0$, proving $Y(t) = X(t)$ for $t \in J$. Since J is arbitrary, $Y = X$ holds on I.
Existence: Let $J := [t_0, a] \subset I$ be a closed interval, and define a sequence $\{X_j\}$ of matrix-valued functions defined on I satisfying $X_0(t) = \mathrm{id}$ and

$$X_{j+1}(t) = \mathrm{id} + \int_{t_0}^{t} X_j(\tau)\Omega(\tau)\,d\tau \quad (j = 0, 1, 2, \dots). \tag{1.11}$$

Let $k := 2\sup_J |\Omega|_{\mathrm{M}}$. Then

$$\begin{aligned} |X_{j+1}(t) - X_j(t)|_{\mathrm{M}} &\leqq \int_{t_0}^{t} |X_j(\tau) - X_{j-1}(\tau)|_{\mathrm{M}}|\Omega(\tau)|_{\mathrm{M}}\,d\tau \\ &\leqq \frac{e^{k(t-t_0)}}{|k|}\sup_J |\Omega|_{\mathrm{M}}||X_j - X_{j-1}||_{J,k} \end{aligned}$$

for an appropriate choice of $k \in \mathbb{R}$, and hence $||X_{j+1} - X_j||_{J,k} \leqq \frac{1}{2}||X_j - X_{j-1}||_{J,k}$, that is, $\{X_j\}$ is a Cauchy sequence with respect to $||\cdot||_{J,k}$. Thus, by completeness (Lemma 1.11), it converges to some $X \in C^0(J, \mathrm{M}_n(\mathbb{R}))$. By (1.11), the limit X satisfies

$$X(t_0) = \mathrm{id}, \qquad X(t) = \mathrm{id} + \int_{t_0}^{t} X(\tau)\Omega(\tau)\,d\tau.$$

Applying the fundamental theorem of calculus, we can see that X satisfies $X'(t) = X(t)\Omega(t)$ ($' = d/dt$). Since J can be taken arbitrarily, existence of the solution on I is proven.

Finally, we shall prove that X is of class C^∞. Since $X'(t) = X(t)\Omega(t)$, the derivative X' of X is continuous. Hence X is of class C^1, and so is $X(t)\Omega(t)$. Thus we have that $X'(t)$ is of class C^1, and then X is of class C^2. Iterating this argument, we can prove that $X(t)$ is of class C^r for arbitrary r. □

Corollary 1.13. *Let $\Omega(t)$ be a matrix-valued C^∞-function defined on an interval I. Then for each $t_0 \in I$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function $X_{t_0,X_0}(t)$ defined on I such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \quad X(t_0) = X_0 \quad \bigl(X(t) := X_{t_0,X_0}(t)\bigr) \tag{1.12}$$

In particular, $X_{t_0,X_0}(t)$ is of class C^∞ in X_0 and t.

Proof. We rewrite $X(t)$ in Proposition 1.12 as $Y(t) = X_{t_0,\mathrm{id}}(t)$. Then the function

$$X(t) := X_0 Y(t) = X_0 X_{t_0,\mathrm{id}}(t), \tag{1.13}$$

is desired one. Conversely, assume $X(t)$ satisfies the conclusion. Noticing $Y(t)$ is a regular matrix for all t because of Proposition 1.8,

$$W(t) := X(t)Y(t)^{-1}$$

satisfies

$$\frac{dW}{dt} = \frac{dX}{dt}Y^{-1} - XY^{-1}\frac{dY}{dt}Y^{-1} = X\Omega Y^{-1} - XY^{-1}Y\Omega Y^{-1} = O.$$

Hence

$$W(t) = W(t_0) = X(t_0)Y(t_0)^{-1} = X_0.$$

Hence the uniqueness is obtained. The final part is obvious by the expression (1.13). □

Proposition 1.14. *Let $\Omega(t)$ and $B(t)$ be matrix-valued C^∞-functions defined on I. Then for each $t_0 \in I$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function defined on I satisfying*

$$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad X(t_0) = X_0. \tag{1.14}$$

Proof. Rewrite X in Proposition 1.12 as $Y := X_{t_0,\mathrm{id}}$. Then

$$X(t) = \left(X_0 + \int_{t_0}^{t} B(\tau)Y^{-1}(\tau)\,d\tau\right)Y(t) \tag{1.15}$$

satisfies (1.14). Conversely, if X satisfies (1.14), $W := XY^{-1}$ satisfies

$$X' = W'Y + WY' = W'Y + WY\Omega, \quad X\Omega + B = WY\Omega + B,$$

and then we have $W' = BY^{-1}$. Since $W(t_0) = X_0$,

$$W = X_0 + \int_{t_0}^{t} B(\tau)Y^{-1}(\tau)\,d\tau.$$

Thus we obtain (1.15). □

Theorem 1.15. *Let I and U be an interval and a domain in $\mathbb{R}^m$, respectively, and let $\Omega(t,\boldsymbol{\alpha})$ and $B(t,\boldsymbol{\alpha})$ be matrix-valued C^∞-functions defined on $I \times U$ ($\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_m)$). Then for each $t_0 \in I$, $\boldsymbol{\alpha} \in U$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0,X_0,\boldsymbol{\alpha}}(t)$ defined on I such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t,\boldsymbol{\alpha}) + B(t,\boldsymbol{\alpha}), \qquad X(t_0) = X_0. \tag{1.16}$$

Moreover,

$$I \times I \times \mathrm{M}_n(\mathbb{R}) \times U \ni (t, t_0, X_0, \boldsymbol{\alpha}) \mapsto X_{t_0,X_0,\boldsymbol{\alpha}}(t) \in \mathrm{M}_n(\mathbb{R})$$

is a C^∞-map.

Proof. Let $\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}}) := \Omega(t+t_0,\boldsymbol{\alpha})$ and $\widetilde{B}(t,\tilde{\boldsymbol{\alpha}}) = B(t+t_0,\boldsymbol{\alpha})$, and let $\widetilde{X}(t) := X(t+t_0)$. Then (1.16) is equivalent to

$$\frac{d\widetilde{X}(t)}{dt} = \widetilde{X}(t)\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}}) + \widetilde{B}(t,\tilde{\boldsymbol{\alpha}}), \quad \widetilde{X}(0) = X_0, \tag{1.17}$$

where $\tilde{\boldsymbol{\alpha}} := (t_0,\alpha_1,\dots,\alpha_m)$. There exists the unique solution $\widetilde{X}(t) = \widetilde{X}_{\mathrm{id},X_0,\tilde{\boldsymbol{\alpha}}}(t)$ of (1.17) for each $\tilde{\boldsymbol{\alpha}}$ because of Proposition 1.14. So it is sufficient to show differentiability with respect to the parameter $\tilde{\boldsymbol{\alpha}}$. We set $Z = Z(t)$ the unique solution of

$$\frac{dZ}{dt} = Z\widetilde{\Omega} + \widetilde{X}\frac{\partial\widetilde{\Omega}}{\partial\alpha_j} + \frac{\partial\widetilde{B}}{\partial\alpha_j}, \qquad Z(0) = O. \tag{1.18}$$

Then it holds that $Z = \partial\widetilde{X}/\partial\alpha_j$ (Problem 1-2). In particular, by the proof of Proposition 1.14, it holds that

$$Z = \frac{\partial\widetilde{X}}{\partial\alpha_j} = \left(\int_0^t \left(\widetilde{X}(\tau)\frac{\partial\widetilde{\Omega}(\tau,\tilde{\boldsymbol{\alpha}})}{\partial\alpha_j} + \frac{\partial\widetilde{B}(\tau,\tilde{\boldsymbol{\alpha}})}{\partial\alpha_j}\right)Y^{-1}(\tau)d\tau\right)Y(t).$$

Here, $Y(t)$ is the unique matrix-valued C^∞-function satisfying $Y'(t) = Y(t)\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}})$, and $Y(0) = \mathrm{id}$. Hence $\widetilde{X}$ is a C^∞-function in $(t,\tilde{\boldsymbol{\alpha}})$. □

Fundamental Theorem for Space Curves. As an application, we prove the fundamental theorem for space curves. A C^∞-map $\gamma\colon I \to \mathbb{R}^3$ defined on an interval $I \subset \mathbb{R}$ into $\mathbb{R}^3$ is said to be a *regular curve* if $\dot\gamma \neq \mathbf{0}$ holds on I. For a regular curve $\gamma(t)$, there exists a parameter change $t = t(s)$ such that $\tilde\gamma(s) := \gamma(t(s))$ satisfies $|\tilde\gamma'(s)| = 1$. Such a parameter s is called the *arc-length parameter*.

Let $\gamma(s)$ be a regular curve in $\mathbb{R}^3$ parametrized by the arc-length satisfying $\gamma''(s) \neq \mathbf{0}$ for all s. Then

$$\boldsymbol{e}(s) := \gamma'(s), \qquad \boldsymbol{n}(s) := \frac{\gamma''(s)}{|\gamma''(s)|}, \qquad \boldsymbol{b}(s) := \boldsymbol{e}(s) \times \boldsymbol{n}(s)$$

forms a positively oriented orthonormal basis $\{\boldsymbol{e}, \boldsymbol{n}, \boldsymbol{b}\}$ of $\mathbb{R}^3$ for each s. Regarding each vector as column vector, we have the matrix-valued function

$$\mathcal{F}(s) := (\boldsymbol{e}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)) \in \mathrm{SO}(3). \tag{1.19}$$

in s, which is called the *Frenet frame* associated to the curve γ. Under the situation above, we set

$$\kappa(s) := |\gamma''(s)| > 0, \qquad \tau(s) := -\left\langle \boldsymbol{b}'(s), \boldsymbol{n}(s) \right\rangle,$$

which are called the *curvature* and *torsion*, respectively, of γ. Using these quantities, the Frenet frame satisfies

$$\frac{d\mathcal{F}}{ds} = \mathcal{F}\varOmega, \qquad \varOmega = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}. \tag{1.20}$$

Proposition 1.16. *The curvature and the torsion are invariant under the transformation $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ of $\mathbb{R}^3$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$). Conversely, two curves $\gamma_1(s)$, $\gamma_2(s)$ parametrized by arc-length parameter have common curvature and torsion, there exist $A \in \mathrm{SO}(3)$ and $\boldsymbol{b} \in \mathbb{R}^3$ such that $\gamma_2 = A\gamma_1 + \boldsymbol{b}$.*

Proof. Let κ, τ and $\mathcal{F}_1$ be the curvature, torsion and the Frenet frame of γ_1, respectively. Then the Frenet frame of $\gamma_2 = A\gamma_1 + \boldsymbol{b}$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$) is $\mathcal{F}_2 = A\mathcal{F}_1$. Hence both $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy (1.20), and then γ_1 and γ_2 have common curvature and torsion.

Conversely, assume γ_1 and γ_2 have common curvature and torsion. Then the frenet frame $\mathcal{F}_1$, $\mathcal{F}_2$ both satisfy (1.20). Let $\mathcal{F}$ be the unique solution of (1.20) with $\mathcal{F}(t_0) = \mathrm{id}$. Then by the proof of Corollary 1.13, we have $\mathcal{F}_j(t) = \mathcal{F}_j(t_0)\mathcal{F}(t)$ ($j = 1, 2$). In particular, since $\mathcal{F}_j \in \mathrm{SO}(3)$, $\mathcal{F}_2(t) = A\mathcal{F}_1(t)$ ($A := \mathcal{F}_2(t_0)\mathcal{F}_1(t_0)^{-1} \in \mathrm{SO}(3)$). Comparing the first column of these, $\gamma_2'(s) = A\gamma_1'(t)$ holds. Integrating this, the conclusion follows. □

Theorem 1.17 (The fundamental theorem for space curves)**.**
Let $\kappa(s)$ and $\tau(s)$ be C^∞-fnctions defined on an interval I satisfying $\kappa(s) > 0$ on I. Then there exists a space curve $\gamma(s)$ parametrized by arc-length whose curvature and torsion are κ and τ, respectively. Moreover, such a curve is unique up to transformation $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$) of $\mathbb{R}^3$.

Proof. We have already shown the uniqueness in Proposition 1.16. We shall prove the existence: Let $\varOmega(s)$ be as in (1.20), and $\mathcal{F}(s)$ the solution of (1.20) with $\mathcal{F}(s_0) = \mathrm{id}$. Since $\varOmega$ is skew-symmetric, $\mathcal{F}(s) \in \mathrm{SO}(3)$ by Proposition 1.10. Denoting the column vectors of $\mathcal{F}$ by $\boldsymbol{e}$, $\boldsymbol{n}$, $\boldsymbol{b}$, and let

$$\gamma(s) := \int_{s_0}^{s} \boldsymbol{e}(\sigma)\, d\sigma.$$

Then $\mathcal{F}$ is the Frenet frame of γ, and κ, and τ are the curvature and torsion of γ, respectively. □

Exercises

1-1 Find the maximal solution of the initial value problem

$$\frac{dx}{dt} = \lambda x(a - x), \qquad x(0) = b,$$

where λ and a are positive constants, and b is a real number.

1-2 Verify that Z in (1.18) coincides with $\partial \widetilde{X}/\partial \alpha_j$.

1-3 Find an explicit expression of a space curve $\gamma(s)$ parametrized by the arc-length s, whose curvature and torsion are $a/(1+s^2)$ and $b/(1+s^2)$, respectively, where a and b are constants.