2 Integrability Conditions

Let $U \subset \mathbb{R}^m$ be a domain of $(\mathbb{R}^m; u^1, \dots, u^m)$ and consider an m-tuple of $n \times n$-matrix valued C^∞-maps

$$\Omega_j \colon \mathbb{R}^m \supset U \longrightarrow \mathrm{M}_n(\mathbb{R}) \qquad (j = 1, \dots, m). \tag{2.1}$$

In this section, we consider an initial value problem of a system of linear partial differential equations

$$\frac{\partial X}{\partial u^j} = X \Omega_j \quad (j = 1, \dots, m), \qquad X(\mathrm{P}_0) = X_0, \tag{2.2}$$

where $\mathrm{P}_0 = (u^1_0, \dots, u^m_0) \in U$ is a fixed point, X is an $n \times n$-matrix valued unknown, and $X_0 \in \mathrm{M}_n(\mathbb{R})$. The chain rule yields the following:

Lemma 2.1. *Let $X \colon U \to \mathrm{M}_n(\mathbb{R})$ be a C^∞-map satisfying (2.2). Then for each smooth path $\gamma \colon I \to U$ defined on an interval $I \subset \mathbb{R}$, $\hat{X} := X \circ \gamma : I \to \mathrm{M}_n(\mathbb{R})$ satisfies the ordinary differential equation*

$$\frac{d\hat{X}}{dt}(t) = \hat{X}(t)\Omega_\gamma(t) \qquad \left(\Omega_\gamma(t) := \sum_{j=1}^m \Omega_j \circ \gamma(t) \frac{du^j}{dt}(t)\right) \tag{2.3}$$

on I, where $\gamma(t) = \bigl(u^1(t), \dots, u^m(t)\bigr)$.

Proposition 2.2. *If a C^∞-map $X \colon U \to \mathrm{M}_n(\mathbb{R})$ defined on a domain $U \subset \mathbb{R}^m$ satisfies (2.2) with $X_0 \in \mathrm{GL}(n, \mathbb{R})$, then $X(\mathrm{P}) \in \mathrm{GL}(n, \mathbb{R})$ for all $\mathrm{P} \in U$. In addition, if Ω_j $(j = 1, \dots, m)$ are skew-symmetric and $X_0 \in \mathrm{SO}(n)$, then $X(\mathrm{P}) \in \mathrm{SO}(n)$ holds for all $\mathrm{P} \in U$.*

Proof. Since U is connected, there exists a continuous path $\gamma_0 \colon [0, 1] \to U$ such that $\gamma_0(0) = \mathrm{P}_0$ and $\gamma_0(1) = \mathrm{P}$. By Whitney's approximation theorem (cf. Theorem 6.21 in [Lee13]), there exists a smooth path $\gamma \colon [0, 1] \to U$ joining P_0 and P approximating γ_0. Since $\hat{X} := X \circ \gamma$ satisfies (2.3) with $\hat{X}(0) = X_0$, Proposition 1.8 yields that $\det \hat{X}(1) \neq 0$ whenever $\det X_0 \neq 0$. Moreover, if Ω_j's are skew-symmetric, so is $\Omega_\gamma(t)$ in (2.3). Thus, by Proposition 1.10, we obtain the latter half of the proposition. □

Proposition 2.3. *If a matrix-valued C^∞ function $X \colon U \to \mathrm{GL}(n, \mathbb{R})$ satisfies (2.2), it holds that*

$$\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j \Omega_k - \Omega_k \Omega_j \tag{2.4}$$

for each (j, k) with $1 \leqq j < k \leqq m$.

Proof. Differentiating (2.2) by u^k, we have

$$\frac{\partial^2 X}{\partial u^k \partial u^j} = \frac{\partial X}{\partial u^k}\Omega_j + X \frac{\partial \Omega_j}{\partial u^k} = X\left(\frac{\partial \Omega_j}{\partial u^k} + \Omega_k \Omega_j\right).$$

On the other hand, switching the roles of j and k, we get

$$\frac{\partial^2 X}{\partial u^j \partial u^k} = X\left(\frac{\partial \Omega_k}{\partial u^j} + \Omega_j \Omega_k\right).$$

Since X is of class C^∞, the left-hand sides of these equalities coincide, and so are the right-hand sides. Since $X \in \mathrm{GL}(n, \mathbb{R})$, the conclusion follows. □

26. April, 2022. (Revised: 10. May, 2022)

The equality (2.4) is called the *integrability condition* or *compatibility condition* of (2.2).

Lemma 2.4. *Let $\Omega_j\colon U \to \mathrm{M}_n(\mathbb{R})$ $(j = 1, \dots, m)$ be C^∞-maps defined on a domain $U \subset \mathbb{R}^m$ which satisfy* (2.4)*. Then for each smooth map*

$$\sigma\colon D \ni (t, w) \longmapsto \sigma(t, w) = (u^1(t, w), \dots, u^m(t, w)) \in U$$

defined on a domain $D \subset \mathbb{R}^2$, it holds that

$$\frac{\partial T}{\partial w} - \frac{\partial W}{\partial t} - TW + WT = 0, \tag{2.5}$$

where

$$T := \sum_{j=1}^{m} \widetilde{\Omega}_j \frac{\partial u^j}{\partial t}, \quad W := \sum_{j=1}^{m} \widetilde{\Omega}_j \frac{\partial u^j}{\partial w} \quad (\widetilde{\Omega}_j := \Omega_j \circ \sigma). \tag{2.6}$$

Proof. By the chain rule, we have

$$\begin{aligned}
\frac{\partial T}{\partial w} &= \sum_{j,k=1}^{m} \frac{\partial \Omega_j}{\partial u^k} \frac{\partial u^k}{\partial w} \frac{\partial u^j}{\partial t} + \sum_{j=1}^{m} \widetilde{\Omega}_j \frac{\partial^2 u^j}{\partial w \partial t}, \\
\frac{\partial W}{\partial t} &= \sum_{j,k=1}^{m} \frac{\partial \Omega_j}{\partial u^k} \frac{\partial u^k}{\partial t} \frac{\partial u^j}{\partial w} + \sum_{j=1}^{m} \widetilde{\Omega}_j \frac{\partial^2 u^j}{\partial t \partial w} \\
&= \sum_{j,k=1}^{m} \frac{\partial \Omega_k}{\partial u^j} \frac{\partial u^j}{\partial t} \frac{\partial u^k}{\partial w} + \sum_{j=1}^{m} \widetilde{\Omega}_j \frac{\partial^2 u^j}{\partial t \partial w}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial T}{\partial w} - \frac{\partial W}{\partial t} &= \sum_{j,k=1}^{m} \left(\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} \right) \frac{\partial u^k}{\partial w} \frac{\partial u^j}{\partial t} \\
&= \sum_{j,k=1}^{m} \left(\widetilde{\Omega}_j \widetilde{\Omega}_k - \widetilde{\Omega}_k \widetilde{\Omega}_j \right) \frac{\partial u^k}{\partial w} \frac{\partial u^j}{\partial t} \\
&= \left(\sum_{j=1}^{m} \widetilde{\Omega}_j \frac{\partial u^j}{\partial t} \right) \left(\sum_{k=1}^{m} \widetilde{\Omega}_k \frac{\partial u^k}{\partial w} \right) - \left(\sum_{k=1}^{m} \widetilde{\Omega}_k \frac{\partial u^k}{\partial w} \right) \left(\sum_{j=1}^{m} \widetilde{\Omega}_j \frac{\partial u^j}{\partial t} \right) \\
&= TW - WT.
\end{aligned}$$

Thus (2.5) holds.

Integrability of linear systems. The main theorem in this section is the following Frobenius' theorem:

Theorem 2.5. *Let $\Omega_j\colon U \to \mathrm{M}_n(\mathbb{R})$ $(j = 1, \dots, m)$ be C^∞-functions defined on a simply connected domain $U \subset \mathbb{R}^m$ satisfying* (2.4)*. Then for each $\mathrm{P}_0 \in U$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique $n \times n$-matrix valued function $X\colon U \to \mathrm{M}_n(\mathbb{R})$ satisfying* (2.2)*. Moreover,*

- *if $X_0 \in \mathrm{GL}(n, \mathbb{R})$, $X(\mathrm{P}) \in \mathrm{GL}(n, \mathbb{R})$ holds on U,*
- *if $X_0 \in \mathrm{SO}(n)$ and Ω_j $(j = 1, \dots, m)$ are skew-symmetric matrices, $X \in \mathrm{SO}(n)$ holds on U.*

Proof. The latter half is a direct conclusion of Proposition 2.2. We show the existence of X: Take a smooth path $\gamma\colon [0, 1] \to U$ joining P_0 and P. Then by Theorem 1.15, there exists a unique C^∞-map $\hat{X}\colon [0, 1] \to \mathrm{M}_n(\mathbb{R})$ satisfying (2.3) with initial condition $\hat{X}(0) = X_0$.

We shall show that the value $\hat{X}(1)$ does not depend on choice of paths joining P_0 and P. To show this, choose another smooth path $\tilde{\gamma}$ joining P_0 and P. Since U is simply connected, there exists a homotopy between γ and $\tilde{\gamma}$, that is, there exists a continuous map $\sigma_0\colon [0,1]\times[0,1] \ni (t,w) \mapsto \sigma(t,w) \in U$ satisfying

$$\begin{aligned} \sigma_0(t,0) &= \gamma(t), \qquad & \sigma_0(t,1) &= \tilde{\gamma}(t), \\ \sigma_0(0,w) &= \mathrm{P}_0, \qquad & \sigma_0(1,w) &= \mathrm{P}. \end{aligned} \tag{2.7}$$

Then, by Whitney's approximation theorem (Theorem 6.21 in [Lee13]) again, there exists a smooth map $\sigma\colon [0,1]\times[0,1] \to U$ satisfying the same boundary conditions as (2.7):

$$\begin{aligned} \sigma(t,0) &= \gamma(t), \qquad & \sigma(t,1) &= \tilde{\gamma}(t), \\ \sigma(0,w) &= \mathrm{P}_0, \qquad & \sigma(1,w) &= \mathrm{P}. \end{aligned} \tag{2.8}$$

We set T and W as in (2.6). For each fixed $w \in [0,1]$, there exists $X_w\colon [0,1] \to \mathrm{M}_n(\mathbb{R})$ such that

$$\frac{dX_w}{dt}(t) = X_w(t)T(t,w), \qquad X_w(0) = X_0.$$

Since $T(t,w)$ is smooth in t and w, the map

$$\check{X}\colon [0,1]\times[0,1] \ni (t,w) \mapsto X_w(t) \in \mathrm{M}_n(\mathbb{R})$$

is a smooth map, because of smoothness in parameter α in Theorem 1.15. To show that $\hat{X}(1) = \check{X}(1,0)$ does not depend on choice of paths, it is sufficient to show that

$$\frac{\partial \check{X}}{\partial w} = \check{X}W \tag{2.9}$$

holds on $[0,1]\times[0,1]$. In fact, by (2.8), $W(1,w) = 0$ for all $w \in [0,1]$, and then (2.9) implies that $\check{X}(1,w)$ is constant.

We prove (2.9): By definition, it holds that

$$\frac{\partial \check{X}}{\partial t} = \check{X}T, \qquad \check{X}(0,w) = X_0 \tag{2.10}$$

for each $w \in [0,1]$. Hence by (2.5),

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial \check{X}}{\partial w} &= \frac{\partial^2 \check{X}}{\partial t \partial w} = \frac{\partial^2 \check{X}}{\partial w \partial t} = \frac{\partial}{\partial w}(\check{X}T) \\ &= \frac{\partial \check{X}}{\partial w}T + \check{X}\frac{\partial T}{\partial w} = \frac{\partial \check{X}}{\partial w}T + \check{X}\left(\frac{\partial W}{\partial t} + TW - WT\right) \\ &= \frac{\partial \check{X}}{\partial w}T + \check{X}\frac{\partial W}{\partial t} + \frac{\partial \check{X}}{\partial t}W - \check{X}WT \\ &= \frac{\partial}{\partial t}(\check{X}W) + \left(\frac{\partial \check{X}}{\partial w} - \check{X}W\right)T. \end{aligned}$$

So, the function $Y_w(t) := \partial\check{X}/\partial w - \check{X}W$ satisfies the ordinary differential equation

$$\frac{dY_w}{dt}(t) = Y_w(t)T(t,w), \quad Y_w(0) = O$$

for each $w \in [0,1]$. Thus, by the uniqueness of the solution, $Y_w(t) = O$ holds on $[0,1]\times[0,1]$. Hence we have (2.9).

Thus, $\hat{X}(1)$ depends only the end point P of the path. Hence we can set $X(\mathrm{P}) := \hat{X}(1)$ for each $\mathrm{P} \in U$, and obtain a map $X\colon U \to \mathrm{M}_n(\mathbb{R})$. Finally we show that X is the desired solution. The initial condition $X(\mathrm{P}_0) = X_0$ is obviously satisfied. On the other hand, if we set

$$Z(\delta) := X(u^1, \dots, u^j + \delta, \dots, u^m),$$

$Z(\delta)$ satisfies the equation (2.3) for the path $\gamma(\delta) := (u^1, \dots, u^j + \delta, \dots, u^m)$ with $Z(0) = X(\mathrm{P})$. Since $\Omega_\gamma = \Omega_j$,

$$\frac{\partial X}{\partial u^j}(\mathrm{P}) = \left.\frac{dZ}{d\delta}\right|_{\delta=0} = Z(0)\Omega_j(\mathrm{P}) = X(\mathrm{P})\Omega_j(\mathrm{P})$$

which completes the proof. □

Application: Poincaré's lemma.

Theorem 2.6 (Poincaré's lemma)**.** *If a differential* 1*-form*

$$\omega = \sum_{j=1}^{m} \alpha_j(u^1, \dots, u^m)\, du^j$$

defined on a simply connected domain $U \subset \mathbb{R}^m$ *is closed, that is,* $d\omega = 0$ *holds, then there exists a* C^∞*-function* f *on* U *such that* $df = \omega$*. Such a function* f *is unique up to additive constants.*

Proof. Since

$$d\omega = \sum_{i<j} \left(\frac{\partial \alpha_j}{\partial u^i} - \frac{\partial \alpha_i}{\partial u^j} \right) du^i \wedge du^j,$$

the assumption is equivalent to

$$\frac{\partial \alpha_j}{\partial u^i} - \frac{\partial \alpha_i}{\partial u^j} = 0 \qquad (1 \leqq i < j \leqq m). \tag{2.11}$$

Consider a system of linear partial differential equations with unknown ξ, a 1×1-matrix valued function (i.e. a real-valued function), as

$$\frac{\partial \xi}{\partial u^j} = \xi \alpha_j \quad (j = 1, \dots, m), \qquad \xi(u_0^1, \dots, u_0^m) = 1. \tag{2.12}$$

Then it satisfies (2.4) because of (2.11). Hence by Theorem 2.5, there exists a smooth function $\xi(u^1, \dots, u^m)$ satisfying (2.12). In particular, Proposition 1.8 yields $\xi = \det \xi$ never vanishes. Hence $\xi(u_0^1, \dots, u_0^m) = 1 > 0$ means that $\xi > 0$ holds on U. Letting $f := \log \xi$, we have the function f satisfying $df = \omega$.

Next, we show the uniqueness: if two functions f and g satisfy $df = dg = \omega$, it holds that $d(f - g) = 0$. Hence by connectivity of U, $f - g$ must be constant. □

Application: Conjugation of Harmonic functions. In this paragraph, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. It is well-known that a function

$$f\colon U \ni u + \mathrm{i}\, v \longmapsto \xi(u, v) + \mathrm{i}\, \eta(u, v) \in \mathbb{C} \qquad (\mathrm{i} = \sqrt{-1}) \tag{2.13}$$

defined on a domain $U \subset \mathbb{C}$ is *holomorphic* if and only if it satisfies the following relation, called the *Cauchy-Riemann equations*:

$$\frac{\partial \xi}{\partial u} = \frac{\partial \eta}{\partial v}, \qquad \frac{\partial \xi}{\partial v} = -\frac{\partial \eta}{\partial u}. \tag{2.14}$$

Definition 2.7. A function $f\colon U \to \mathbb{R}$ defined on a domain $U \subset \mathbb{R}^2$ is said to be *harmonic* if it satisfies

$$\Delta f = f_{uu} + f_{vv} = 0.$$

The operator Δ is called the *Laplacian*.

Proposition 2.8. *If function f in* (2.13) *is holomorphic, $\xi(u,v)$ and $\eta(u,v)$ are harmonic functions.*

Proof. By (2.14), we have

$$\xi_{uu} = (\xi_u)_u = (\eta_v)_u = \eta_{vu} = \eta_{uv} = (\eta_u)_v = (-\xi_v)_v = -\xi_{vv}.$$

Hence $\Delta\xi = 0$. Similarly,

$$\eta_{uu} = (-\xi_v)_u = -\xi_{vu} = -\xi_{uv} = -(\xi_u)_v = -(\eta_v)_v = -\eta_{vv}.$$

Thus $\Delta\eta = 0$. □

Theorem 2.9. *Let $U \subset \mathbb{C} = \mathbb{R}^2$ be a simply connected domain and $\xi(u,v)$ a C^∞-function harmonic on U*[6]*. Then there exists a C^∞ harmonic function η on U such that $\xi(u,v) + \mathrm{i}\,\eta(u,v)$ is holomorphic on U.*

Proof. Let $\alpha := -\xi_v\,du + \xi_u\,dv$. Then by the assumption,

$$d\alpha = (\xi_{vv} + \xi_{uu})\,du \wedge dv = 0$$

holds, that is, α is a closed 1-form. Hence by simple connectivity of U and the Poincaré's lemma (Theorem 2.6), there exists a function η such that $d\eta = \eta_u\,du + \eta_v\,dv = \alpha$. Such a function η satisfies (2.14) for given ξ. Hence $\xi + \mathrm{i}\,\eta$ is holomorphic in $u + \mathrm{i}\,v$. □

Example 2.10. A function $\xi(u,v) = e^u \cos v$ is harmonic. Set

$$\alpha := -\xi_v\,du + \xi_u\,dv = e^u \sin v\,du + e^u \cos v\,dv.$$

Then $\eta(u,v) = e^u \sin v$ satisfies $d\eta = \alpha$. Hence

$$\xi + \mathrm{i}\,\eta = e^u(\cos v + \mathrm{i} \sin v) = e^{u+\mathrm{i}\,v}$$

is holomorphic in $u + \mathrm{i}\,v$.

Definition 2.11. The harmonic function η in Theorem 2.9 is called the *conjugate* harmonic function of ξ.

Exercises

2-1 Let $\xi(u,v) := \log\sqrt{u^2+v^2}$ be a function defined on $U := \mathbb{R}^2 \setminus \{(0,0)\}$

(1) Show that ξ is harmonic on U.

(2) Find the conjugate harmonic function η of ξ on

$$V = \mathbb{R}^2 \setminus \{(u,0) \mid u \leqq 0\} \subset U.$$

(3) Show that there exists no conjugate harmonic function of ξ defined on U.

[6] The theorem holds under the assumption of C^2-differentiablity.

2-2 Let $\theta = \theta(u,v)$ be a smooth function on a domain $U \subset \mathbb{R}^2$ such that $0 < \theta < \pi$, and set

$$\Omega(u,v) := \begin{pmatrix} \theta_u \cot\theta & 0 & \cot\theta \\ -\theta_u \csc\theta & 0 & -\csc\theta \\ 0 & \sin\theta & 0 \end{pmatrix}, \quad \Lambda(u,v) := \begin{pmatrix} 0 & -\theta_v \csc\theta & -\csc\theta \\ 0 & \theta_v \cot\theta & \cot\theta \\ \sin\theta & 0 & 0 \end{pmatrix}.$$

Prove that the compatibility condition of a system of partial differentail equation

$$\frac{\partial \mathcal{F}}{\partial u} = \mathcal{F}\Omega, \qquad \frac{\partial \mathcal{F}}{\partial v} = \mathcal{F}\Lambda$$

is equivalent to

$$\theta_{uv} = \sin\theta.$$

2-3 Let $\boldsymbol{v} = \boldsymbol{v}(x,y,z)$ be a vector field defined on a simply connected domain U in $(\mathbb{R}^3; (x,y,z))$. Assume that $\boldsymbol{v}$ is *irrotational*, that is, $\operatorname{rot}\boldsymbol{v} = \boldsymbol{0}$. Then there exists a function $\varphi\colon U \to \mathbb{R}$ such that $\boldsymbol{v} = \operatorname{grad}\varphi$.