3 A review of surface theory

In this section, we review the classical surface theory in the Euclidean 3-space. The textbook [UY17] is one of the fundamental references of this material.

3.1 Preliminaries

Euclidean space Let $\mathbb{R}^3$ be the Euclidean 3-space, that is, the 3-dimensional affine space $\mathbb{R}^3$ endowed with the Euclidean *inner product* "$\cdot$", where[7]

$$\boldsymbol{x}\cdot\boldsymbol{y} := {}^t\boldsymbol{x}\boldsymbol{y} = x^1y^1 + x^2y^2 + x^3y^3, \qquad \text{where} \quad \boldsymbol{x} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} \in \mathbb{R}^3. \tag{3.1}$$

The Euclidean *norm* $|\ |$ and the Euclidean *distance* $d(\ ,\ )$ is defined as

$$|\boldsymbol{x}| := \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}} = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}, \qquad d(\boldsymbol{x},\boldsymbol{y}) = |\boldsymbol{y}-\boldsymbol{x}| \qquad (\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3). \tag{3.2}$$

A map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ is called an *isometry* if it preserves the distance function d: $d\bigl(f(\boldsymbol{x}), f(\boldsymbol{y})\bigr) = d(\boldsymbol{x},\boldsymbol{y})$ ($\boldsymbol{x}$, $\boldsymbol{y}\in\mathbb{R}^3$).

Fact 3.1. *A map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry if and only if f is in a form*

$$f(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b} \qquad (A \in \mathrm{O}(3), \boldsymbol{b}\in\mathbb{R}^3), \tag{3.3}$$

where $\mathrm{O}(3)$ is the set of 3×3 orthogonal matrices.

An isometry in (3.3) is said to be *orientation preserving* if $A \in \mathrm{SO}(3)$, that is, A is an orthogonal matrix with $\det A = 1$.

The *outer product* or *vector product* of $\boldsymbol{x}\times\boldsymbol{y}$ of $\boldsymbol{x}$, $\boldsymbol{y}\in\mathbb{R}^3$ is defined by

$$\det(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = (\boldsymbol{x}\times\boldsymbol{y})\cdot\boldsymbol{z}. \tag{3.4}$$

Immersed surfaces Let $U \subset \mathbb{R}^2$ be a domain of the uv-plane $\mathbb{R}^2$. A C^∞-map $p\colon U \to \mathbb{R}^3$ is called an *immersion* or a *parametrization of a regular surface* if

$$p_u(u,v) := \frac{\partial p}{\partial u}(u,v), \quad \text{and} \quad p_v(u,v) := \frac{\partial p}{\partial v}(u,v) \qquad \text{are linearly independent} \tag{3.5}$$

at each point $(u,v)\in U$. The *unit normal vector field* to an immersion $p\colon U\to\mathbb{R}^3$ is a C^∞-map $\nu\colon U \to \mathbb{R}^3$ satisfying

$$\nu\cdot p_u = \nu\cdot p_v = 0, \qquad |\nu| = 1 \tag{3.6}$$

for each point on U.

The *first fundamental form* ds^2 is defined by

$$ds^2 := dp\cdot dp = E\,du^2 + 2F\,du\,dv + G\,dv^2, \qquad \bigl(E := p_u\cdot p_u, F := p_u\cdot p_v = p_v\cdot p_u, G := p_v\cdot p_v\bigr), \tag{3.7}$$

where the subscript u (resp. v) means the partial derivative with respect to the variable u (resp. v). The three functions E, F and G defined on U are called the coefficients of the first fundamental form.

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[7] According to a traditional manner, the indices of coordinate functions are written as superscripts.

Similarly, taking account of the identity

$$\nu_u \cdot p_v = (\nu \cdot p_v)_u - \nu \cdot p_{vu} = 0 - \nu \cdot p_{vu} = -\nu \cdot p_{uv} = \nu_v \cdot p_u,$$

we define the *second fundamental form* as

$$\text{(3.8)} \quad I\!I := -d\nu \cdot dp = L\,du^2 + 2M\,du\,dv + N\,dv^2, \quad \bigl(L := -p_u \cdot \nu_u, M := -p_u \cdot \nu_v = -p_v \cdot \nu_u, N := -p_v \cdot \nu_v\bigr).$$

The symmetric matrices

$$\widehat{I} := \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} {}^t p_u \\ {}^t p_v \end{pmatrix} (p_u, p_v), \qquad \widehat{I\!I} := \begin{pmatrix} L & M \\ M & N \end{pmatrix} = -\begin{pmatrix} {}^t p_u \\ {}^t p_v \end{pmatrix} (\nu_u, \nu_v)$$

are called the first and second fundamental matrices, respectively.

By the Cauchy-Schwartz inequality, it holds that

$$EG - F^2 = |p_u|^2 |p_v|^2 - (p_u \cdot p_v)^2 > 0,$$

and then the first fundamental matrix $\widehat{I}$ is a regular matrix. The *area element* of the surface is defined as

$$\text{(3.9)} \qquad d\mathcal{A} := \sqrt{EG - F^2}\,du\,dv.$$

In fact, the area of a part of surface corresponding to a relatively compact domain $\Omega \subset U$ is computed as

$$\mathcal{A}(\Omega) := \iint_\Omega d\mathcal{A} = \iint_\Omega \sqrt{EG - F^2}\,du\,dv.$$

Since $\widehat{I}$ is regular, the matrix

$$\text{(3.10)} \qquad A := \widehat{I}^{-1}\,\widehat{I\!I} = \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix},$$

called the *Weingarten matrix*, is defined. It is known that the eigenvalues λ_1 and λ_2 of A are real numbers, and called the *principal curvatures*. The *Gaussian curvature* K and the *mean curvature* H are defined as

$$\text{(3.11)} \qquad K := \lambda_1 \lambda_2 = \det A = \frac{\det \widehat{I\!I}}{\det \widehat{I}}, \qquad H := \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{2}\operatorname{tr} A.$$

3.2 Gauss frames

To simplify computations, and for a future generalization for higher dimensional case, we switch the notation here to the "index" style. Write the coordinate system of $U \subset \mathbb{R}^2$ by (u^1, u^2) instead of (u, v), and denote

$$f_{,1} = \frac{\partial f}{\partial u^1}, \qquad f_{,2} = \frac{\partial f}{\partial u^2},$$

that is, the subscript number following a comma means the partial derivative with respect to the corresponding variable. Using these notations, the first fundamental form is expressed as

$$\text{(3.12)} \qquad ds^2 = dp \cdot dp = \sum_{i,j=1}^{2} g_{ij}\,du^i\,du^j, \qquad (g_{ij} := p_{,i} \cdot p_{,j}).$$

Similarly, the second fundamental form is written as

$$II = -dp \cdot d\nu = \sum_{i,j=1}^{2} h_{ij}\, du^i\, du^j, \qquad (h_{ij} := -p_{,i} \cdot \nu_{,j} = -p_{,j} \cdot \nu_{,i} = p_{,ij} \cdot \nu). \tag{3.13}$$

Since the first fundamental matrix $\widehat{I} = (g_{ij})_{i,j=1,2}$ has positive determinant, its inverse matrix exists. We denote the component of the inverse by $\widehat{I}^{-1} = (g^{ij})$, using superscripts instead of subscripts. By definition, it hold that

$$g^{ij} = g^{ji} \qquad \text{and} \qquad \sum_{k=1}^{2} g^{ik} g_{kj} = \delta^i_j = \begin{cases} 1 & (i = j) \\ 0 & (\text{otherwise}), \end{cases} \tag{3.14}$$

where δ stands for *Kronecker's delta symbol*. Using these, the Weingarten matrix A as in (3.10) and the Gaussian curvature K in (3.11) are expressed as

$$A = (A^j_i), \qquad A^j_i = \sum_{k=1}^{2} g^{jk} h_{ki}, \qquad K = \det A = \frac{\det(h_{ij})}{\det(g_{ij})}. \tag{3.15}$$

Since p is an immersion, $\{p_{,1}(u^1, u^2), p_{,2}(u^1, u^2), \nu(u^1, u^2)\}$ are linearly independent for each point $(u^1, u^2) \in U$. Hence we obtain a smooth map

$$\mathcal{F}\colon U \ni (u^1, u^2) \mapsto \bigl(p_{,1}(u^1, u^2), p_{,2}(u^1, u^2), \nu(u^1, u^2)\bigr) \in \mathrm{GL}(3, \mathbb{R}), \tag{3.16}$$

where $\mathrm{GL}(3, \mathbb{R})$ is the set of 3×3 regular matrices with real components. The map $\mathcal{F}$ is called the *Gauss frame* of the surface p.

Theorem 3.2. *The Gauss frame $\mathcal{F}$ satisfies*

$$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\Omega_j \qquad \left(\Omega_j := \begin{pmatrix} \Gamma^1_{1j} & \Gamma^1_{2j} & -A^1_j \\ \Gamma^2_{1j} & \Gamma^2_{2j} & -A^2_j \\ h_{1j} & h_{2j} & 0 \end{pmatrix}\right) \qquad (j = 1, 2), \tag{3.17}$$

where h_{ij}'s are the coefficients of the second fundamental form, A^i_j's are the components of the Weingarten matrix, and

$$\Gamma^k_{ij} := \frac{1}{2} \sum_{l=1}^{2} g^{kl} (g_{il,j} + g_{lj,i} - g_{ij,l}), \qquad (i, j, k = 1, 2) \tag{3.18}$$

The functions Γ^k_{ij} in (3.18) are called the *Christoffel symbols*, and the equation (3.17) is called the *Gauss-Weingarten formula*. By decomposing $\mathcal{F}$ into columns, the Gauss-Weingarten formula is restated as

$$p_{,ij} = \left(\sum_{l=1}^{2} \Gamma^l_{ij} p_{,l}\right) + h_{ij}\nu, \tag{3.19}$$

$$\nu_{,j} = -\sum_{l=1}^{2} A^l_j p_{,l}. \tag{3.20}$$

The equality (3.19) and (3.20) are called the *Gauss formula* and *Weingarten formula*, respectively.

Proof of Theorem 3.2. Since $\{p_{,1}, p_{,2}, \nu\}$ is a basis of $\mathbb{R}^3$ at each point $(u^1, u^2) \in U$, the second derivative $p_{,ij}$ is expressed as a linear combination of $\{p_{,1}, p_{,2}, \nu\}$:

$$p_{,ij} = \Lambda^1_{ij} p_{,1} + \Lambda^2_{ij} p_{,2} + \eta_{ij}\nu = \left(\sum_{l=1}^{2} \Lambda^l_{ij} p_{,l}\right) + \eta_{ij}\nu, \tag{3.21}$$

where Λ^l_{ij} and η_{ij} are smooth functions in (u^1,u^2). Since ν is perpendicular to $p_{,l}$, (3.13) implies

$$\eta_{ij} = p_{,ij}\cdot\nu = h_{ij}.$$

On the other hand, taking inner product with $p_{,k}$, we have

$$p_{,ij}\cdot p_{,k} = \sum_{l=1}^{2}\Lambda^l_{ij}p_{,l}\cdot p_{,k} = \sum_{l=1}^{2} g_{lk}\Lambda^l_{ij}. \tag{3.22}$$

Here, by the Leibniz rule, the left-hand side is computed as

$$\begin{aligned}p_{,ij}\cdot p_{,k} &= (p_{,i}\cdot p_{,k})_{,j} - p_{,i}\cdot p_{,kj} = g_{ik,j} - (p_{,i}\cdot p_{,j})_{,k} + p_{,ik}\cdot p_{,j}\\ &= g_{ik,j} - g_{ij,k} + (p_{,k}\cdot p_{,j})_{,i} - p_{,ij}\cdot p_{,k} = g_{ik,j} - g_{ij,k} + g_{jk,i} - p_{,ij}\cdot p_{,k},\end{aligned}$$

and thus, $p_{,ij}\cdot p_{,k} = \frac{1}{2}(g_{ik,j}+g_{kj,i}-g_{ij,k})$. Then (3.22) turns to be

$$\frac{1}{2}(g_{ik,j}+g_{kj,i}-g_{ij,k}) = p_{,ij}\cdot p_{,k} = \sum_{l=1}^{2} g_{lk}\Lambda^l_{ij}.$$

Multiplying g^{sk} on the both side of the equality above, and summing up it over $k=1$ and 2, we have

$$\frac{1}{2}\sum_{k=1}^{2} g^{sk}(g_{ik,j}+g_{kj,i}-g_{ij,k}) = \sum_{k=1}^{2}\sum_{l=1}^{2} g^{sk}g_{lk}\Lambda^l_{ij} = \sum_{l=1}^{2}\sum_{s=1}^{2} g^{sk}g_{kl}\Lambda^l_{ij} = \sum_{l=1}^{2}\delta^s_l\Lambda^l_{ij} = \Lambda^s_{ij}.$$

This implies that Λ^l_{ij} coincides with the Christoffel symbol (3.18). Summing up, the Gauss formula (3.19) is proven.

Next, we prove the Weingarten formula: Since $\nu\cdot\nu = 1$, $\nu_{,j}$ is perpendicular to ν. Hence we can write

$$\nu_{,j} = \sum_{l=1}^{2} B^l_j p_{,l},$$

and then by (3.21),

$$-h_{ij} = p_{,i}\cdot\nu_{,j} = \sum_{l=1}^{2} B^l_j p_{,l}\cdot p_{,i} = \sum_{l=1}^{2} g_{il}B^l_j.$$

So,

$$B^k_j = \sum_{l=1}^{2}\delta^k_l B^l_j = \sum_{l=1}^{2}\sum_{s=1}^{2} g^{ks}g_{sl}B^l_j = -\sum_{s=1}^{2} g^{ks}h_{js} = -A^k_j,$$

proving (3.20). □

For later use, we prepare the following formulas on the Christoffel symbols:

Proposition 3.3. *The Christoffel symbol in* (3.18) *satisfies*

$$\Gamma^k_{ij} = \Gamma^k_{ji} \tag{3.23}$$

$$g_{ij,k} = \sum_{l=1}^{2}(g_{lj}\Gamma^l_{ik} + g_{il}\Gamma^l_{kj}), \tag{3.24}$$

$$\frac{\partial g}{\partial u^i} = 2g\sum_{l=1}^{2}\Gamma^l_{il}, \qquad (g := \det\widehat{I} = g_{11}g_{22} - g_{12}^2), \tag{3.25}$$

where the indices i, j and k run over 1 *and* 2*.*

Proof. Since

$$p_{,ij} = \Gamma^1_{ij} p_{,1} + \Gamma^2_{ij} p_{,2} + h_{ij}\nu \quad \text{and} \quad p_{,ji} = \Gamma^1_{ji} p_{,1} + \Gamma^2_{ji} p_{,2} + h_{ji}\nu,$$

(3.23) follows.

The second formula (3.24) is obtained as

$$\begin{aligned} g_{ij,k} &= \bigl(p_{,i}\cdot p_{,j}\bigr)_{,k} = p_{,ik}\cdot p_{,j} + p_{,i}\cdot p_{,jk} \\ &= \left(\sum_{l=1}^{2} \Gamma^l_{ik}(p_{,l}\cdot p_{,j}) + h_{ik}(\nu\cdot p_{,j})\right) + \left(\sum_{l=1}^{2} \Gamma^l_{jk}(p_{,i}\cdot p_{,l}) + h_{jk}(p_{,i}\cdot \nu)\right) \\ &= \sum_{l=1}^{2} \left(g_{lj}\Gamma^l_{ik} + g_{il}\Gamma^l_{kj}\right). \end{aligned}$$

Finally, differentiating $g = \det \widehat{I}$,

$$\begin{aligned} \frac{\partial g}{\partial u^i} &= \operatorname{tr}\left(\widetilde{\widehat{I}} \frac{\partial \widehat{I}}{\partial u^i}\right) = (\det \widehat{I}) \operatorname{tr}\left(\widehat{I}^{-1} \widehat{I}_{,i}\right) = g \sum_{l,m=1}^{2} g^{lm} g_{lm,i} \\ &= g \sum_{l,m,s=1}^{2} g^{lm}\left(g_{ms}\Gamma^s_{li} + g_{ls}\Gamma^s_{im}\right) = g\left(\sum_{l,s=1}^{2} \delta^l_s \Gamma^s_{li} + \sum_{m,s=1}^{2} \delta^m_s \Gamma^s_{im}\right) \\ &= g\left(\sum_{l=1}^{2} \Gamma^l_{li} + \sum_{m=1}^{2} \Gamma^m_{im}\right) = 2g \sum_{l=1}^{2} \Gamma^l_{il}, \end{aligned}$$

where $\widetilde{\widehat{I}} = (\det \widehat{I})\, \widehat{I}^{-1}$ is the cofactor matrix of $\widehat{I}$. Thus we have (3.25). □

3.3 Orthonormal frames

The Gauss and Weingarten formulas (Theorem 3.2) are the fundamental equations which express how the fundamental forms determine shape of surfaces. In this section, another formulation of Gauss-Weingarten formulas using orthonormal frames. In this subsection, we write the coordinate system of $\mathbb{R}^2$ by (u, v), again.

Adapted frames

Let $p\colon U \to \mathbb{R}^3$ be an immersion of a domain $U \subset \mathbb{R}^2$ into the Euclidean 3-space, and take the unit normal vector field $\nu\colon U \to \mathbb{R}^3$ of p. For a simplicity, we assume that ν is compatible to the canonical orientation of U, that is, $\det \mathcal{F} = \det(p_u, p_v, \nu) > 0$, where $\mathcal{F}$ is the Gauss frame.

Definition 3.4. A C^∞-map $\mathcal{E} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) : U \to \mathrm{SO}(3)$ is called an *adapted* (orthonormal) frame of the surface $p\colon U \to \mathbb{R}^3$ if $\boldsymbol{e}_3$ coincides with the unit normal vector field ν.

Example 3.5. Let $p\colon \mathbb{R}^2 \supset U \ni (u, v) \mapsto p(u, v) \in \mathbb{R}^3$ be an immersion and let ν be the unit normal vector field of p which is compatible to the orientation of U. We let

$$\boldsymbol{e}^0_1 := \frac{1}{\sqrt{E}} p_u, \qquad \boldsymbol{e}^0_2 := \frac{1}{\sqrt{E}\sqrt{EG - F^2}}(E p_v - F p_u),$$

where E, F, G are the coefficients of the first fundamental form as in (3.7). Since $\nu := \boldsymbol{e}^0_3$ is perpendicular to both p_u and p_v, $\mathcal{E}^0 := (\boldsymbol{e}^0_1, \boldsymbol{e}^0_2, \boldsymbol{e}^0_3)$ is an adapted frame of p. Remark that $\{\boldsymbol{e}^0_1, \boldsymbol{e}^0_2\}$ is an orthonormal frame of the orthogonal complement of ν (that is, the tangent plane) obtained by applying the Gram-Schmidt orthogonalization to (p_u, p_v).

Gauge transformations

An adapted frame has an ambiguity of a rotation of the frame $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ of the tangent plane. In fact, for an arbitrary function $\phi\colon U \to \mathbb{R}$,

$$\widetilde{\mathcal{E}} = \mathcal{E}R, \qquad R := R_\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.26}$$

is another adapted frame. Conversely, we have the following:

Lemma 3.6. *Let $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ be adapted frames of the surface $p\colon U \to \mathbb{R}^3$, where U is a simply connected domain. Then there exists a function $\phi\colon U \to \mathbb{R}$ satisfying* (3.26)*.*

Proof. Since $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are valued in SO(3) with common third columns, an SO(3)-valued function $R := \mathcal{E}^{-1}\widetilde{\mathcal{E}}$ is expressed as

$$R = \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} R_0 & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix}, \qquad \left(R_0 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : U \to \mathrm{SO}(2) \right),$$

where a and b are C^∞-functions defined on U. Fix a point $(u_0, v_0) \in U$. Since $R_0 \in \mathrm{SO}(2)$, $a^2 + b^2 = 1$, and then there exists an angle ϕ_0 such that

$$a(u_0, v_0) = \cos\phi_0, \quad b(u_0, v_0) = \sin\phi_0. \tag{3.27}$$

Consider a differential 1-form

$$\omega := -b\,da + a\,db = (-ba_u + ab_u)\,du + (-ba_v + ab_v)\,dv.$$

Then

$$d\omega = \bigl((-ba_v + ab_v)_u - (-ba_u + ab_u)_v\bigr)\,du \wedge dv = 2(a_u b_v - b_u a_v)\,du \wedge dv.$$

On the other hand, differentiating $a^2 + b^2 = 1$, it holds that

$$0 = a\,da + b\,db = (aa_u + bb_u)du + (aa_v + bb_v)dv, \qquad \text{that is,} \qquad aa_u = -bb_u, \qquad aa_v = -bb_v.$$

Hence

$$\begin{aligned} a d\omega &= 2(aa_u b_v - b_u a a_v)\,du \wedge dv = 2(-bb_u b_v + b_u a a_v)\,du \wedge dv = 0, \\ b d\omega &= 2(a_u b b_v - b b_u a_v)\,du \wedge dv = 2(-a_u a a_v + a a_u a_v) du \wedge dv = 0, \end{aligned}$$

which implies that $d\omega = 0$ because $(a, b) \neq (0, 0)$. Then by the Poincaré lemma (Theorem 2.6), there exists the unique function $\phi\colon U \to \mathbb{R}$ such that

$$d\phi = \omega = -b\,da + a\,db, \qquad \phi(u_0, v_0) = \phi_0. \tag{3.28}$$

Set $\tilde{a} := \cos\phi$ and $\tilde{b} := \sin\phi$. Then by (3.28), both R_0 and

$$\hat{R}_0 = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

satisfies the same systems of differential equations

$$X_u = X \begin{pmatrix} 0 & -\phi_u \\ \phi_u & 0 \end{pmatrix}, \qquad X_v = X \begin{pmatrix} 0 & -\phi_v \\ \phi_v & 0 \end{pmatrix}$$

with the same initial condition. Hence $R_0 = \hat{R}_0$, which is the conclusion. □

A transformation of adapted frames as in Lemma 3.6 is called a *gauge transformation*.

Gauss-Weingarten formulas

Let $\mathcal{E} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be an adapted frame of a surface $p\colon U \to \mathbb{R}^3$. Since $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are perpendicular to ν, there exists a matrix

$$\check{I} = \begin{pmatrix} g_1^1 & g_2^1 \\ g_1^2 & g_2^2 \end{pmatrix} \qquad \text{such that} \qquad (p_u, p_v) = (\boldsymbol{e}_1, \boldsymbol{e}_2)\,\check{I}\,. \tag{3.29}$$

On the other hand, since $\boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1$, the derivatives of $\boldsymbol{e}_3$ are perpendicular to $\boldsymbol{e}_3$. Then there exists a matrix $\check{II}$ such that

$$\check{II} = \begin{pmatrix} h_1^1 & h_2^1 \\ h_1^2 & h_2^2 \end{pmatrix} \qquad \text{such that} \qquad \bigl((\boldsymbol{e}_3)_u, (\boldsymbol{e}_3)_v\bigr) = -(\boldsymbol{e}_1, \boldsymbol{e}_2)\,\check{II}\,. \tag{3.30}$$

Lemma 3.7. *The Gaussian curvature K satisfy*

$$K = \frac{\det \check{II}}{\det \check{I}}$$

Proof. The first and second fundamental matrices are

$$\begin{aligned} \widehat{I} &= \begin{pmatrix} {}^t p_u \\ {}^t p_v \end{pmatrix} (p_u, p_v) = {}^t\,\check{I} \begin{pmatrix} {}^t\boldsymbol{e}_1 \\ {}^t\boldsymbol{e}_2 \end{pmatrix} (\boldsymbol{e}_1, \boldsymbol{e}_2)\,\check{I} = ({}^t\,\check{I}\,)\,\check{I}\,, \\ \widehat{II} &= -\begin{pmatrix} {}^t p_u \\ {}^t p_v \end{pmatrix} (\nu_u, \nu_v) = {}^t\,\check{I} \begin{pmatrix} {}^t\boldsymbol{e}_1 \\ {}^t\boldsymbol{e}_2 \end{pmatrix} (\boldsymbol{e}_1, \boldsymbol{e}_2)\,\check{II} = ({}^t\,\check{I}\,)\,\check{II}\,. \end{aligned}$$

Hence we have the conclusion by (3.11). □

Proposition 3.8. *There exist functions α, β defined on U such that*

$$\mathcal{E}_u = \mathcal{E}\Omega, \qquad \mathcal{E}_v = \mathcal{E}\Lambda \qquad \left(\Omega := \begin{pmatrix} 0 & -\alpha & -h_1^1 \\ \alpha & 0 & -h_1^2 \\ h_1^1 & h_1^2 & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} 0 & -\beta & -h_2^1 \\ \beta & 0 & -h_2^2 \\ h_2^1 & h_2^2 & 0 \end{pmatrix} \right). \tag{3.31}$$

Proof. Since $\mathcal{E}$ is SO(3)-valued, $\Omega := \mathcal{E}^{-1}\mathcal{E}_u$ and $\Lambda := \mathcal{E}^{-1}\mathcal{E}_v$ are skew-symmetric matrices. The third columns of Ω and Λ are nothing but the definition of the matrix $\check{II}$. □

Definition 3.9. The differential form

$$\mu := \alpha\,du + \beta\,dv$$

is called the *connection form* with respect to the adapted frame.

Lemma 3.10. *The connection forms μ and $\tilde{\mu}$ of the adapted frames $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ as in Lemma 3.6 satisfy*

$$\tilde{\mu} = \mu + d\phi.$$

Proof. Let $\widetilde{\Omega} := \widetilde{\mathcal{E}}^{-1}\widetilde{\mathcal{E}}_u$ and $\widetilde{\Lambda} := \widetilde{\mathcal{E}}^{-1}\widetilde{\mathcal{E}}_v$. Then

$$\widetilde{\Omega} = \widetilde{\mathcal{E}}^{-1}(\mathcal{E}_u R + \mathcal{E}R_u) = \widetilde{\mathcal{E}}^{-1}(\mathcal{E}\Omega R + \mathcal{E}R_u) = \widetilde{\mathcal{E}}^{-1}\widetilde{\mathcal{E}}(R^{-1}\Omega R + R^{-1}R_u) = R^{-1}\Omega R + R^{-1}R_u,$$

and $\widetilde{\Lambda} = R^{-1}\Lambda R + R^{-1}R_u$ hold. Then the conclusion follows. □

Exercises

3-1 Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ are given in the form

$$ds^2 = e^{2\sigma}((du^1)^2 + (du^2)^2), \qquad I\!I = \sum_{i,j=1}^{2} h_{ij}\, du^i\, du^j,$$

where σ is a smooth function in (u^1, u^2). Compute the matrices Ω_j $(j = 1, 2)$ in (3.17).

3-2 Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ are given in the form

$$ds^2 = (du^1)^2 + 2\cos\theta\, du^1\, du^2 + (du^2)^2, \qquad I\!I = 2\sin\theta\, du^1\, du^2,$$

where θ is a smooth function in (u^1, u^2). Compute the matrices Ω_j $(j = 1, 2)$ in (3.17).