# 6 Pseudospherical surfaces

A *pseudospherical surface* is a surface with constant negative Gaussian curvature. In this section, we introduce a way to construct pseudospherical surfaces via the fundamental theorem for surface theory. Throughout this section, we let $p\colon U \to \mathbb{R}^3$ be a regular surface defined on a domain $U$ of the $uv$-plane and let $\nu$ be the unit normal vector field. Denote by $ds^2$ and $I\!I$ the first and second fundamental forms:

$$ds^2 = dp \cdot dp = E\,du^2 + 2\,F\,du\,dv + G\,dv^2, \qquad I\!I = -dp \cdot d\nu = L\,du^2 + 2\,M\,du\,dv + N\,dv^2.$$

By a homothety in $\mathbb{R}^3$, it is sufficient to consider only the case of Gaussian curvature $-1$.

## 6.1 Asymptotic coordinate systems

A non-zero tangent vector $\boldsymbol{a} := \alpha p_u(u_0, v_0) + \beta p_v(u_0, v_0)$ of the surface at $\mathrm{P} = p(u_0, v_0)$ ($(u_0, v_0) \in U$) is called an *asymptotic vector* if

$$\alpha^2 L(u_0, v_0) + 2\alpha\beta M(u_0, v_0) + \beta^2 N(u_0, v_0) = 0,$$

and the subspace $[\boldsymbol{a}]$ in the tangent space $dp(T_{(u_0,v_0)}U)$ spanned by $\boldsymbol{a}$ is called the *asymptotic direction* at $(u_0, v_0)$.

**Lemma 6.1.** *Assume that the Gaussian curvature $K$ is negative at $(u_0, v_0)$. Then there exists a neighborhood $V$ of $(u_0, v_0)$ and smooth functions $\alpha_i$, $\beta_i$ ($i = 1, 2$) on $V$ such that*

$$\boldsymbol{\alpha}_i(u, v) := \alpha_i(u, v)p_u(u, v) + \beta_i(u, v)p_v(u, v) \qquad (i = 1, 2) \tag{6.1}$$

*are two linearly independent asymptotic directions at each $(u, v) \in V$.*

*Proof.* Since the Gaussian curvature is negative at $(u_0, v_0)$, the determinant of the second fundamental matrix $\widehat{I\!I}$ is negative on some neighborhood $V$ of $(u_0, v_0)$, that is, there exists an orthogonal matrix-valued function $P = P(u, v)$ and positive function $\lambda_i(u, v)$ ($i = 1, 2$) defined on $V$ such that

$$\widehat{I\!I} = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = {}^t P \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} P \tag{6.2}$$

holds on $V$. Then

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = P^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = P^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are desired functions. □

Let $\gamma(t) = (u(t), v(t))$ be a regular curve on $U$ and denote $\hat\gamma = p \circ \gamma$. Then $\hat\gamma$ is called an *asymptotic curve* if the velocity vector $\dot{\hat\gamma}(t)$ is an asymptotic direction for each $t$. In other words,

**Lemma 6.2.** *A curve $\hat\gamma = p \circ \gamma$ is an asymptotic curve if and only if*

$$\dot u(t)^2 L(u(t), v(t)) + 2\dot u(t)\dot v(t) M(u(t), v(t)) + \dot v(t)^2 N(u(t), v(t)) = 0$$

*for all $t$, where $\gamma(t) = (u(t), v(t))$ is a regular curve on $U$.*

**Definition 6.3.** A parameter $(u, v)$ of the surface $p\colon U \to \mathbb{R}^3$ is called an *asymptotic coordinate system* or an *asymptotic parameter* if both the $u$-curves $u \mapsto p(u, v)$ and the $v$-curves $v \mapsto p(u, v)$ are asymptotic curves.

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**Lemma 6.4.** *A coordinate system $(u,v)$ of a surface is an asymptotic coordinate system if and only if the second fundamental form is written in the form*

$$II = 2M\,du\,dv,$$

*that is, $L = N = 0$. In particular, $M \neq 0$ if the Gaussian curvature does not vanish.*

*Proof.* Since the $u$-curve $u \mapsto p(u,v)$ is an asymptotic curve for each fixed $v$, Lemma 6.2 implies that $L = 0$. Similarly, the $v$-curve $v \mapsto p(u,v)$ is also an asymptotic curve, we have $N = 0$. Since $K = -M^2/(EG - F^2)$, we have the last assertion. □

**Corollary 6.5.** *Assume both $(u,v)$ and $(\xi,\eta)$ are asymptotic coordinate systems of a surface with negative Gaussian curvature. Then the coordinate change $(\xi,\eta) \mapsto (u,v)$ is in one of the following forms:*

$$(u,v) = (u(\xi), v(\eta)), \qquad (u,v) = (u(\eta), v(\xi)),$$

*where $u(\xi)$ etc. are smooth functions in one variable whose derivatives never vanish.*

*Proof.* By Lemma 6.4, the second fundamental form of the surface is written as

$$II = 2M\,du\,dv = 2\widetilde{M}\,d\xi\,d\eta.$$

On the other hand, by the chain rule, it holds that

$$II = 2M\,(u_\xi\,d\xi + u_\eta\,d\eta)(v_\xi\,d\xi + v_\eta, d\eta) = 2M(u_\xi v_\xi\,d\xi^2 + (u_\xi v_\eta + u_\eta v_\xi)\,d\xi\,d\eta + u_\eta v_\eta\,d\eta^2).$$

Thus, we have

$$u_\xi v_\xi = 0 \qquad \text{and} u_\eta v_\eta = 0.$$

Here, noticing that the coordinate change must satisfy $u_\xi v_\eta - u_\eta v_\xi \neq 0$, one of the following holds:

$$\begin{aligned} u_\eta = v_\xi = 0, &\quad \text{and} \quad u_\xi v_\eta \neq 0,\\ u_\xi = v_\eta = 0, &\quad \text{and} \quad u_\eta v_\xi \neq 0. \end{aligned}$$

This completes the proof. □

**Theorem 6.6.** *Let $p\colon U \to \mathbb{R}^3$ be a regular surface whose Gaussian curvature at $(u_0,v_0)$ is negative. Then there exists a neighborhood $V'$ of $(u_0,v_0)$ and a coordinate change $V' \ni (\xi,\eta) \mapsto (u(\xi,\eta), v(\xi,\eta)) \in U$ for which $(\xi,\eta)$ is an asymptotic coordinate system.*

*Proof.* This proof is due to [KN63], and also found in [UY17, Appendix B-5].

By Lemma 6.1, there exist smooth functions $\alpha_j$, $\beta_j$ $(j = 1,2)$ on $U$ such that the asymptotic directions are given by (6.1) for each $(u,v)$. We set

$$\boldsymbol{v}_j = \boldsymbol{v}_j(u,v) := \begin{pmatrix}\alpha_j(u,v)\\ \beta_j(u,v)\end{pmatrix} \qquad (j = 1,2),$$

which are two linearly independent vector fields on $U$.

Take a regular curve $\sigma_1(s)$ on $U$ with

$$\sigma_1(0) = (u_0,v_0) \qquad \text{and} \qquad \frac{d}{ds}\sigma_1(0) \text{ is linearly independent to } \boldsymbol{v}_2(u_0,v_0).$$

Then one can take a smooth map

$$(s,t) \longmapsto \boldsymbol{u}(s,t) = \bigl(u(s,t), v(s,t)\bigr) \tag{6.3}$$

such that
$$\frac{\partial \boldsymbol{u}}{\partial t}(s,t) = \boldsymbol{v}_2(u(s,t),v(s,t)), \qquad \boldsymbol{u}(0,0)=(u_0,v_0),$$
by solving an initial value problem of an ordinary differential equation
$$\frac{d}{dt}\boldsymbol{u}(s,t) = \boldsymbol{\alpha}_2 \circ \boldsymbol{u}(s,t), \qquad \boldsymbol{u}(s,0)=\sigma_1(s)$$
for each $s$. Since
$$\frac{\partial \boldsymbol{u}}{\partial s}(0,0) = \frac{d}{ds}\sigma_1(0) \qquad \text{and} \qquad \frac{\partial \boldsymbol{u}}{\partial t}(0,0) = \boldsymbol{v}_2(u_0,v_0)$$
are linearly independent, the inverse function theorem yields that there exists an inverse map
$$V \ni (u,v) \longmapsto (s(u,v),t(u,v))$$
of (6.3) defined on a sufficiently small neighborhood of $(u_0,v_0)$. Note that the curve $s=$ constant corresponds to the asymptotic curve tangent to $\boldsymbol{\alpha}_2$.

Similarly, for a curve $\sigma_2(\xi)$ with
$$\sigma_2(0)=(u_0,v_0) \qquad \text{and} \qquad \frac{d}{d\xi}\sigma_2(0) \text{ is linearly independent to } \boldsymbol{v}_1(u_0,v_0),$$
and take a smooth map
$$(\xi,\eta) \longmapsto \boldsymbol{u}(\xi,\eta) = (u(\xi,\eta),v(\xi,\eta))$$
such that
$$\frac{\partial \boldsymbol{u}}{\partial \eta}(\xi,\eta) = \boldsymbol{v}_1(u(\xi,\eta),u(\xi,\eta)), \qquad \boldsymbol{u}(0,0)=(u_0,v_0).$$
Then there exists the inverse $(u,v) \mapsto (\xi,\eta)$ defined on a neighborhood of $V$ of $(u_0,v_0)$. By definition, the curve $\eta=$ constant corresponds to the asymptotic curve tangent to $\boldsymbol{\alpha}_1$.

Using these functions, we let
$$\varphi\colon V \ni (u,v) \mapsto (s(u,v),\eta(u,v)),$$
then $(s,\eta)$ is a coordinate system such that both $s$-curves and $\eta$-curves correspond to asymptotic curves. □

### *6.2 Asymptotic Chebyshev net*

**Theorem 6.7.** *Let $p\colon U \to \mathbb{R}^3$ be a surface with Gaussian curvature $-1$. Then for each point $(u_0,v_0)\in U$, there exists a neighborhood $V$ and coordinate change $(\xi,\eta)\mapsto(u,v)$ on $V$ such that the first and second fundamental forms are in the form*
$$ds^2 = d\xi^2 + 2\cos\theta\, d\xi\, d\eta + d\eta^2, \quad II = 2\sin\theta\, d\xi\, d\eta,$$
*where $\theta=\theta(\xi,\eta)$ is a smooth function in $(\xi,\eta)$ valued in $(0,\pi)$.*

*Proof.* By Theorem 6.6, we may assume that $(u,v)$ is an asymptotic coordinate system without loss of generality. Then by Lemma 6.4, the first and second fundamental forms are expressed as
$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2, \qquad II = 2M\,du\,dv.$$
Since the Gaussian curvature is constant, it hold that $E_v=0$ and $G_u=0$, as seen in Exercise 4-1. Hence $E$ (resp. $G$) depends only on the variable $u$ (resp. $v$). Since $E>0$ and $G>0$, we can choose new coordinate system $(\xi,\eta)$ satisfying
$$\xi(u) := \int \sqrt{E(u)}\,du, \qquad \eta(v) := \int \sqrt{G(v)}\,dv.$$

Then the first and second fundamental form are expressed as

$$ds^2 = d\xi^2 + 2\widetilde{F}\,d\xi\,d\eta + d\eta^2, \qquad II = 2\widetilde{M}\,d\xi\,d\eta.$$

Here, the regularity of the surface yields $1 - \widetilde{F}^2 > 0$, that is, there exists a smooth function $\theta = \theta(\xi,\eta) \in (0,\pi)$ such that $\widetilde{F} = \cos\theta$. Moreover, since Gaussian curvature is $-1$, $\widetilde{M}^2 = 1 - \widetilde{F}^2 = \sin^2\theta$ holds. So, by a parameter change $(\xi,\eta) \mapsto (\xi,-\eta)$ and $\theta \mapsto \pi - \theta$, if necessary, we obtain $\widetilde{M} = \sin\theta$. □

*Remark* 6.8. The coordinate system in Theorem 6.7 is called the *asymptotic Chebyshev net*. The asymptotic Chebyshev net is unique up to coordinate changes

$$(u,v) \mapsto (v,u) \qquad \text{and} \qquad (u,v) \mapsto (\pm u + u_0, \pm v + v_0) \qquad (u_0,v_0) \in \mathbb{R}^2,$$

see Corollary 6.5.

### *6.3 Sine-Gordon equation*

As seen in Exercise 5-2, we have

**Theorem 6.9.** *Let $U$ be a domain on $uv$-plane and $\theta\colon U \to (0,\pi)$ a smooth function satisfying*

$$\theta_{uv} = \sin\theta. \tag{6.4}$$

*If $U$ is simply connected, there exists a regular surface $p\colon U \to \mathbb{R}^3$ with first and second fundamental forms as*

$$ds^2 = du^2 + 2\cos\theta\,du\,du + dv^2, \qquad II = 2\sin\theta\,du\,dv.$$

*In particular, $p$ is a pseudospherical surface.*

*Remark* 6.10. The equation (6.4) is called the *sine-Gordon* equation.

There are numerous “famous” solutions of the sine-Gordon equation, we shall find the simplest non-trivial solutions in this section: Assume the function $\theta$ is written in the form $\theta(u,v) = \varphi(u-v)$, where $\varphi = \varphi(t)$ is a smooth function in one variable. Then the sine-Gordon equation (6.4) is reduced to the equation of motion for a pendulum:

$$\ddot{\varphi} = -\sin\varphi. \tag{6.5}$$

The equation has the first integral

$$\frac{1}{2}\dot{\varphi}^2 - \cos\varphi = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\left(1 - 2\sin^2\frac{\varphi}{2}\right).$$

That is, the value

$$e := \left(\frac{\dot{\varphi}}{2}\right)^2 + \sin^2\frac{\varphi}{2} \tag{6.6}$$

is constant along the solution. Now we assume $e = 1$. In this case, (6.6) is rewritten as

$$\frac{\dot{\varphi}}{2} = \pm\cos\frac{\varphi}{2},$$

then one can complete the integration.

### *Exercises*

**6-1** Find an explicit solution of (6.5) for $e = 1$, with initial condition $\varphi(0) = 0$, $\dot{\varphi}(0) = 2$.

**6-2** For a constant $e \in (0,1)$, the solution $\varphi$ of (6.5) with (6.6) is a periodic function. Find the period of such a solution.