Advanced Topics in Geometry F (MTH.B502)

Riemannian metrics

Kotaro Yamada

kotaro@math.titech.ac.jp

http://www.math.titech.ac.jp/~kotaro/class/2022/geom-~~e~~f/

Tokyo Institute of Technology

2022/06/14

Manifold

our interest : local theory.

- M: a differentiable (C^∞) n-manifold ($M \subset \mathbb{R}^n$ domain)
- $(U, x^1, \ldots, x^n)$: a local coordinate system (chart)
- $\boxed{\mathcal{F}(M):}$ the algebra of differentiable functions on M

"vector sp" (∞-dim)

Must consider coordinate change

 接空間

$X: \mathcal{F}(M) \to \mathbb{R}$ linear.

$X(fg) = f(p)\, Xg + (Xf)\, g(p)$

X

- Differential operators on $\mathcal{F}(M)$ at p.
- $\mathrm{Span}\left\{ \left(\frac{\partial}{\partial x^1}\right)_p, \left(\frac{\partial}{\partial x^2}\right)_p, \dots, \left(\frac{\partial}{\partial x^2}\right)_p \right\}$. $= T_pM$ n
- the set of velosity vectors of curves. (速度)

$$\boxed{\dim T_pM = n = \dim M}$$

(Coordinate Change)

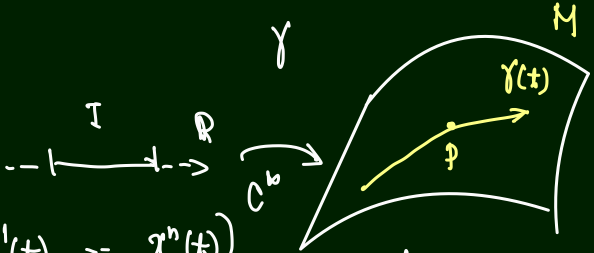
$$\left(\frac{\partial}{\partial x^j}\right)_p = \sum_{k=1}^{n} \frac{\partial y^k}{\partial x^j}(p) \left(\frac{\partial}{\partial y^k}\right)_p$$

chain rule

$(x^1 \cdots x^n) \mapsto (y^1 \cdots y^n)$

$$X = \sum_{j=1}^{n} X^j \left(\frac{\partial}{\partial x^j}\right)_p = \sum_{k=1}^{n} \tilde{X}^k \left(\frac{\partial}{\partial y^k}\right)_p$$

$$\Rightarrow \qquad \tilde{X}^k = \sum_{j=1}^{n} \frac{\partial y^k}{\partial x^j}(p) X^j.$$



$$\gamma(t) = \left(x^1(t), \ldots, x^n(t)\right)$$

$$\gamma(0) = p$$

$$\Rightarrow \quad \dot{\gamma}(0) = \dot{x}^1(0)\left(\frac{\partial}{\partial x^1}\right)_p + \cdots + \dot{x}^n(0)\left(\frac{\partial}{\partial x^n}\right)_p$$

velocity

vect. sp

- the dual of T_pM
- $\mathrm{Span}\{(dx^1)_p, \ldots, (dx^n)_p\}$, dual basis of $\left\{\left(\frac{\partial}{\partial x^i}\right)_p\right\}$

$T_p^*M = \{\alpha : T_pM \to \mathbb{R},\ \text{linear}\}$

$(\dim T_p^*M = n)$

Chain rule

$$(dy^k)_p = \sum_{j=1} \frac{\partial y^k}{\partial x^j}(p)(dx^j)_p. \qquad (1)$$

$$\left((dx^j)_p \left(\left(\frac{\partial}{\partial x^k} \right)_p \right) \right) = \delta^j_k \quad \text{Kronecker's delta}$$

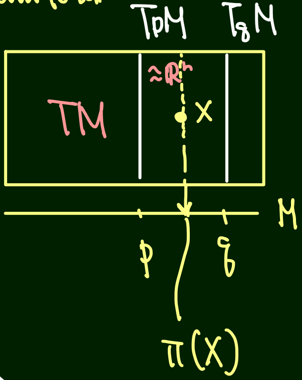
$\Big(T_pM \simeq T_p^*M$: as vector spaces

$\nexists$ canonical isomorphism. (in general) $\Big)$

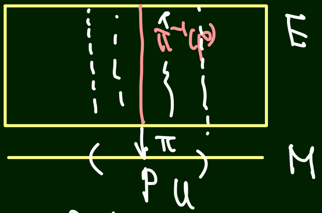
(Rem) $\exists$ canonical isomorphism between T_pM & T_p^*M if a Riem. metric is given

 接束

- $\cancel{M}$ $TM := \bigsqcup_{p\in M} T_pM$ (direct sum) : 2n-dim manifold
- $\pi: TM \to M$: the canonical projection



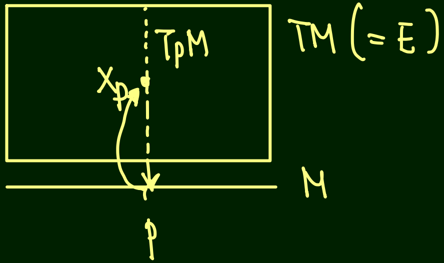
A Vector bundle (ベクトル束)

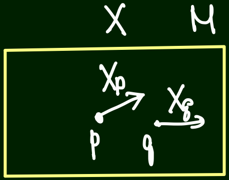


- π: surjection
- $\pi^{-1}(p) \approx F$: a vect sp
- $\pi^{-1}(U) \approx U \times F$

ベクトル場

- $X\colon M \xrightarrow{C^\infty} \boxed{TM}$, $\pi \circ X = \mathrm{id}_M$.
- $\mathfrak{X}(M) = \underbrace{\Gamma(TM)}_{\text{section}}$: the set of vector fields.





$p \longmapsto X_p \in T_pM$ (smooth)

$\mathfrak{X}(M)$: · a vector space ∞-dim

· $\mathfrak{F}(M)$-module ($X \mapsto fX$) · $(fX)_p = f(p)X_p$

Other vector bundles

- T^*M: the cotangent bundle
- $T^*M \otimes T^*M$ $\rightsquigarrow$ $\bigcup_{p \in M} T_p^*M \otimes T_p^*M$
- $S(T^*M \otimes T^*M)$

  symmetri