Advanced Topics in Geometry F (MTH.B502)

Riemannian metrics

Kotaro Yamada
kotaro@math.titech.ac.jp
http://www.math.titech.ac.jp/~kotaro/class/2022/geom-e/

Tokyo Institute of Technology

2022/06/14

Manifold

- M: a differentiable n-manifold
- $(U, x^1, \ldots, x^n)$: a local coordinate system (chart)
- $\mathcal{F}(M)$: the algebra of differentiable functions on M

Tangent space T_pM

- Differential operators on $\mathcal{F}(M)$ at p.
- $\mathrm{Span}\left\{ \left(\frac{\partial}{\partial x^1}\right)_p, \left(\frac{\partial}{\partial x^2}\right)_p, \dots, \left(\frac{\partial}{\partial x^2}\right)_p \right\}$.
- the set of velosity vectors of curves.

$$\left(\frac{\partial}{\partial x^j}\right)_p = \sum_{k=1}^{n} \frac{\partial y^k}{\partial x^j}(p) \left(\frac{\partial}{\partial y^k}\right)_p .$$

$$X = \sum_{j=1}^{n} X^j \left(\frac{\partial}{\partial x^j}\right)_p = \sum_{k=1}^{n} \tilde{X}^k \left(\frac{\partial}{\partial y^k}\right)_p$$

$$\Rightarrow \qquad \tilde{X}^k = \sum_{j=1}^{n} \frac{\partial y^k}{\partial x^j}(p) X^j.$$

Cotangent space T_p^*M

- the dual of T_pM
- $\mathrm{Span}\{(dx^1)_p, \ldots, (dx^n)_p\}$; dual basis

$$(dy^k)_p = \sum_{j=1}^{n} \frac{\partial y^k}{\partial x^j}(p)(dx^j)_p. \tag{1}$$

Tangent bundle TM

- $M := \bigcup_{p \in M} T_pM$
- $\pi\colon TM \to M$: the canonical projection

Vector fields

- $X\colon M \to TM$, $\pi \circ X = \mathrm{id}_M$.
- $\mathfrak{X}(M) = \Gamma(TM)$: the set of vector fields.

Other vector bundles

- T^*M: the cotangent bundle
- $T^*M \otimes T^*M$
- $S(T^*M \otimes T^*M)$