Advanced Topics in Geometry F (MTH.B502)

Riemannian metrics

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2022/06/14

Riemannian manifold

Pseudo Riemannian manifold.

擬リーマン多様体

Definition

A Riemannian metric (resp. pseudo Riemannian metric) is a section $g \in \Gamma(S(T^*M \otimes T^*M))$ such that the quadratic form g_p on T_pM is positive definite inner product (resp. non-degenerate inner product) on T_pM. A pair (M, g) of a manifold M and a (pseudo) Riemannian metric g is called a (pseudo) Riemannian manifold.

manifold

Riem. manifold : $(\underline{\underline{M}}, g)$

$g \in \Gamma(S(T^*M \otimes T^*M))$

"smooth in p"

g_p : an innerproduct of T_pM

内積

Review $\quad V$: a vector space $(\dim V < \infty)$ $\mathbb{R}$

双線型

$$V^* \otimes V^* := \{\beta : V \times V \to \mathbb{R}\ ;\ \text{bilinear}\}$$

$$\left(\left.\begin{array}{l} \beta(X, \cdot) : V \to \mathbb{R} \\ \beta(\cdot, Y) : V \to \mathbb{R} \end{array}\right\} : \text{linear} \right)$$

symmetric

$$S(V^* \otimes V^*) := \{\beta \in V^* \otimes V^*\ ;\ \beta(X, Y) = \beta(Y, X)\}$$

g is · non degenerate $\iff \left(g(X, Y) = 0 \text{ for } \forall Y \Rightarrow X = 0 \right)$ fix

非退化

· positive definite $\iff g(X, X) > 0$ if $X \neq 0$

正定値

Def An inner product on V :

- $g \in S(V^* \otimes V^*)$
- non-degenerate
  or positive definite ← undergraduate lin alg

$V = \mathbb{R}^n = \{\text{column vectors}\}$ $(n \times 1)$ $\ni X, Y$ $\quad g(X, Y) = {}^tXY$

Euclidean inner product (: positive def.

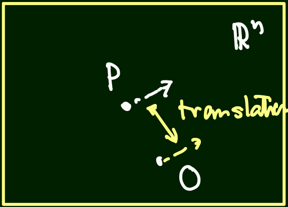
$V = \mathbb{R}^{n+1} \ni X, Y$ $\quad g(X, Y) = {}^tX \begin{bmatrix} -1 & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} Y$

$X = \begin{bmatrix} x^0 \\ x^1 \\ \vdots \\ x^n \end{bmatrix}$ $\quad g(X, X) = -(x^0)^2 + (x^1)^2 + \cdots + (x^n)^2$ non-degenerate

(Minkowski inner product) not positive definite

Example—the Euclidean spaces as Riemannian manifolds

$$\mathbb{R}^n = \{{}^t(x^1, \dots, x^n) ;\ x^i \in \mathbb{R}\}$$



$$T_p\mathbb{R}^n \approx \mathbb{R}^n$$

$$\left(\frac{\partial}{\partial x^i}\right)_p = \begin{array}{cc} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{c} \\ \\ \\ (i \\ \\ \\ \\ \end{array} \end{array}$$

$$X = \begin{bmatrix} X^1 \\ \vdots \\ X^n \end{bmatrix} = \sum X^i \left(\frac{\partial}{\partial x^i}\right)_p \qquad Y = \sum Y^i \left(\frac{\partial}{\partial x^i}\right)_p$$

$$(g_0)_p(X, Y) = \sum X^i Y^i = {}^tXY$$

$(\mathbb{R}^n, g_0)$: the Euclidean space.

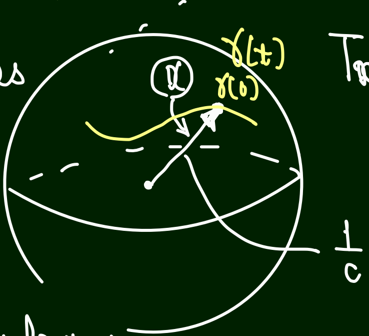
(sectional curvature) $k = c^2 > 0$

$$S^n(k) := \{ x \in \mathbb{R}^{n+1} \,;\, \langle x, x \rangle = \frac{1}{k} \}$$

$\langle , \rangle$ the Euclidean inner product

the sphere of radius $\frac{1}{c}$, centered at the origin (a submanifold in $\mathbb{R}^{n+1}$)

$\langle , \rangle$ induces an inner product on $T_x S^n(k)$ (positive definite)



$T_x S^n(k) = \{ X \in \mathbb{R}^{n+1} \,;\, \langle x, X \rangle = 0 \}$

$\dot\gamma(0) \perp \gamma(0)$

$(\because)\ |\gamma|^2 = \frac{1}{c^2}$

lin. subspace $\boxed{\subset}\ T_x \mathbb{R}^{n+1} = \mathbb{R}^{n+1}$

$$\mathbb{R}^{n+1}_1 = (\mathbb{R}^{n+1}, \langle\,,\,\rangle) \qquad \langle X, Y\rangle = {}^tX \begin{bmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots\; 1 \end{bmatrix} Y$$

(non-deg)

(the Lorentz-Minkowski space as a pseudo Riemannian manifold)

metric is non-pos.-def.

(Rim. mfd : with positive definite metric)

Example—hyperbolic spaces $k = -c^2 < 0$ 双曲空间

$$H^n(k) = H^n(-c^2)$$

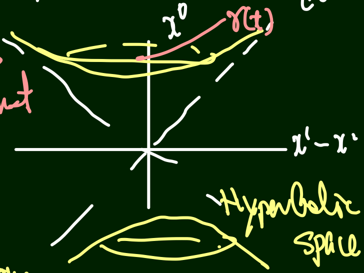
$$x := {}^t[x^0, \dots, x^n]$$

$$= \{ x \in \mathbb{R}^{n+1}_1 \,;\, \langle x, x \rangle = \frac{1}{k},\ cx^0 > 0 \}$$

$$-x_0^2 + x_1^2 + \dots + x_n^2 = -\frac{1}{c^2}$$

$$T_x H^n(k) = x^\perp = \{ X \,;\, \langle X, x \rangle = 0 \}$$

$\langle \gamma(t), \gamma(t) \rangle = \text{const}$



Fact $\langle\ ,\ \rangle|_{T_x H^n(k)}$: positive definite $\rightsquigarrow$ $H^n(k)$: Riem mfd

Exercise 1-1

Problem (Ex. 1-1)

$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = g_{ij} : U \to \mathbb{R}$

Let $U \subset \mathbb{R}^n$ be a domain and g a Riemannian metric on U. Show that

1. *There exists an n-tuple of vector fields $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ such that*

Gram-Schmidt

δ_{ij}

$$g(\boldsymbol{e}_i, \boldsymbol{e}_j) = \delta - ij = \begin{cases} 1 & (i = j) \\ 0 & (\textit{otherwise}) \end{cases}.$$

2. *Take another n-tuple $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ satisfying $g(\boldsymbol{v}_i, \boldsymbol{v}_j) = \delta_{ij}$. Then there exists a matrix-valued function*

$$\Theta : U \xrightarrow{C^\infty} \mathrm{O}(n) \qquad [\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n] = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]\Theta.$$

θ

Gauge Transf.

Exercise 1-2

Problem (Ex. 1-2)

Let $\mathbb{R}^{n+1}_1 = (\mathbb{R}^{n+1}, \langle\ ,\ \rangle_L)$ *be the Minkowski vector space. Show that if* $\boldsymbol{v} \in \mathbb{R}^{n+1}_1$ *satisfies* $\langle \boldsymbol{v}, \boldsymbol{v}\rangle = -1$*, the orthogonal complement*

$$\boldsymbol{v}^{\perp} := \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1\,;\, \langle \boldsymbol{v}, \boldsymbol{x}\rangle_L = 0\}$$

is an n*-dimensional space-like subspace of* $\mathbb{R}^{n+1}_1$*.*

$\langle\ ,\ \rangle|_{v^\perp} > 0$

$\langle v, X\rangle = 0$ $\quad \langle v, v\rangle = -1$

i.e. $X \in v^\perp \setminus \{0\} \Rightarrow \langle X, X\rangle > 0$ $\quad \langle \vec{X}, \vec{X}\rangle \geq 0$

$$X = \begin{bmatrix} x^0 \\ x^1 \\ \vdots \\ x^n \end{bmatrix} = x^0 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x^1 \\ \vdots \\ x^n \end{bmatrix} \quad \vec{X}$$

$x^0 \neq 0$

$\langle\ ,\ \rangle$: positive